2nd Annual Acadia University Undergraduate Mathematics Competition
Time Limit: 2hrs

Solutions to some of the problems

The next three problems are each worth 10 points

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1. The median of five positive integers is 15, the mode is 6 and the mean is 12. What are the five numbers?

   Since the median is 15, two of the numbers must be larger than 15 and two smaller. The *mode* is the most commonly occuring number. As the mode is 6, we now know that three of the numbers are $6, 6, 15$. Since the average is 12, the sum of the five numbers must be 60 and thus the sum of the two as-yet unknown numbers must be 33. Both of these have to be integers larger than 15, so they must be 16 and 17.

   So, the five numbes are $6, 6, 15, 16, 17$.

2. A child on a pogo stick jumps 1 foot on the first jump, 2 feet on the second jump, 4 feet on the third jump, and so on, in general 2^{n-1} feet on the nth jump. Can the child get back to the starting point by a judicious choice of directions?

   The first jump is a distance of 1 foot. All the rest are multiples of 2. There is no way that a sum or difference of multiples of 2 can equal 1, so there is no way to reverse the first jump. Thus, it is impossible for the child to return to the starting point.

3. Find a polynomial $P(x)$ so that $P(x)$ is divisible by $x^2 + 1$ and $P(x) + 1$ is divisible by $x^3 + x^2 + 1$

   We have $P(x) = q(x)(x^2 + 1)$ and $P(x) + 1 = r(x)(x^3 + x^2 + 1)$. Now, $x^3 + x^2 + 1 = (x + 1)(x^2 + 1) - x$, so we get

$$q(x)(x^2+1)+1 = P(x)+1 = r(x)(x^3+x^2+1) = r(x)[(x+1)(x^2+1)-x] = r(x)(x+1)(x^2+1)-xr(x).$$

   So we have

$$(x^2 + 1)[r(x)(x + 1) - q(x)] = 1 + xr(x)$$

and thus $xr(x) + 1$ must be a multiple of $x^2 + 1$. The simplest way for this to happen is if $r(x) = x$ so $1+xr(x) = 1+x^2$. Plugging this in, we see that

$$P(x) = x^4 + x^3 + x - 1,$$

which satisfies the two conditions.

The next four problems are each worth 15 points

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4. Consider a six-sided die which is to be labelled with the numbers $1-6$. We must have 1 and 6 on opposite sides, 2 and 5 on opposite sides and 3 and 4 on opposite sides. Consider two ways of labelling the die as equivalent if you can rotate the die to make the first labelling match the second. In how many nonequivalent ways can the die be labelled?

Since two labellings are equivalent if we can rotate one of them to make it into the other, we will just think about rotating the cube into a *standard configuration*. This is sometimes a useful idea.

So, no matter what the labelling, we can rotate the cube so that 1 is on top, and thus 6 is on the bottom. Now, the numbers $2, 3, 4, 5$ must be around the ‘‘vertical’’ faces. Rotate the cube so that 2 is in the front. Then 5 must be on the back. This sequence of operations can be done NO MATTER WHAT THE LABELLING. So, we see now that there are only two possibilities left. Either 3 is on the left face or it is on the right face.

Thus, there are only two inequivalent labellings of the cube.

5. Prove that at any party, there are two people who have the same number of friends present at the party.

To do this one, you need to assume that if a is a friend with b, then b is also a friend with a. Under these assumptions each person can have $0, 1, 2, \ldots, N-1$ friends at the party, where N is the total number of people at the party.

Suppose that there are not two people at the party with the same number of friends. Label the people at the party $p_1, p_2, \ldots, p_N$ and say that the number of friends for person p_i is n_i. We must have that

$$\{n_1, n_2, \ldots, n_N\} = \{0, 1, 2, 3, \ldots, N-1\}$$

since this is the only way to have N distinct numbers of friends (notice that this doesn't mean that person p_1 must have 0 friends, just that these two sets are the same). However, then this means that one person is friends with everyone and one person is friends with no one, which is impossible.

6. Let M be a 3×3 matrix with all entries drawn randomly (and with equal probability) from $\{0, 1\}$. What is the probability that $det(M)$ will be odd?

For this one, the idea of EVEN versus ODD is very useful, just as in the second problem.

The determinant of a 3×3 matrix $[a_{i,j}]$ is given by

$$a_{1,1}a_{2,2}a_{3,3}+a_{1,2}a_{2,3}a_{3,1}+a_{1,3}a_{2,1}a_{3,2}-a_{1,1}a_{2,3}a_{3,2}-a_{1,2}a_{2,1}a_{3,3}-a_{1,3}a_{2,2}a_{3,1}$$

where each term in the sum is either 0 or 1 (since each $a_{i,j}$ is either 0 or 1).

Now, adding, subtracting, multiplying, dividing even and odd numbers we get the following rules:

+	E	O
E	E	O
O	O	E

×	E	O
E	E	E
O	E	O

Those who have seen modular arithmetic will recognize this as addition and multiplication modulo 2. So, it is enough to think about all our arithmetic modulo 2. The usual notation for the integers modulo 2 is $\mathbb{Z}_2$. So, we have a 3×3 matrix with entries from $\mathbb{Z}_2$. Having a matrix whose determinant is odd corresponds to having a matrix whose determinant is 1 (in this arithmetic), and thus to a matrix which is invertible.

So, how do you construct an invertible matrix? Well, you need to make the matrix have independent columns. To do this, we can choose the first column in any way we like, as long as it isn't the zero vector. So, there are $2^3-1=7$ choices for the first column.

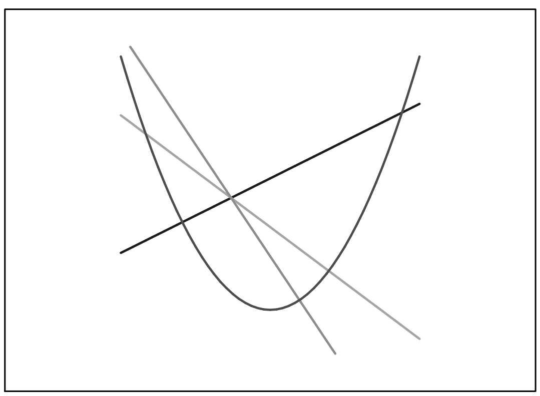
For the second column we have to choose a non-zero vector which is not a multiple of the first. In this case, this just means that it can't EQUAL the first. So, we have 6 choices for this second column, having already chosen the first column.

Finally, for the last column we must choose a non-zero vector which is independent of both the first two columns. Again, in this case this just means that it can't equal either column and can't equal the sum of the first two columns. Thus we have 4 choices for this $(8-1-1-1-1)$.

So, in all there are $7\times 6\times 4 = 168$ different 3×3 matrices with odd determinant. So, the probability of having a matrix with an odd determinant is

$$\frac{168}{512}.$$

This is not the most elementary way of doing this problem. There is another way by splitting into cases and counting each case. This way uses some nice modular arithmetic and some properties of determinants of matrices.



7. Suppose $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) are three points on the parabola $ay = x^2$ which have the property that their normal lines intersect in a common point. Prove that $y_1 + y_2 + y_3 = 0$.

The next three problems are each worth 20 points

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8. Sum the infinite series

$$\frac{3}{1\cdot 2\cdot 3} + \frac{5}{2\cdot 3\cdot 4} + \frac{7}{3\cdot 4\cdot 5} + \cdots$$

9. Suppose it started to snow one morning at a constant rate. At 8:00 AM a snowplow leaves the garage and by 9:00 AM is had gone 2 miles. By 10:00 am it had gone 3 miles. Assuming the snowplow removes a constant volume of snow per hour, show that it had started to snow $(\sqrt{5}-1)/2$ hours ago (or 37 min, 5s).

10. Shuffle an ordinary deck of 52 cards. On average, how far from the top will the first Queen be?

The straightforward way of doing this problem is by using conditional probabilities. The probability that the first Queen is on the ith card from the top (counting the top card as the zeroth) is

$$P_i := \left(\frac{48}{52}\right)\left(\frac{47}{51}\right)\cdots\left(\frac{48-i+1}{52-i+1}\right)\left(\frac{4}{52-i}\right) = \left(\frac{4}{52-i}\right)\prod_{j=1}^{i}\left(\frac{49-j}{53-j}\right).$$

Then the expected distance from the top is

$$0P_0+1P_1+2P_2+\cdots+47P_{47}+48P_{48} = \sum_{i=0}^{48} i\left(\frac{4}{52-i}\right)\prod_{j=1}^{i}\left(\frac{49-j}{53-j}\right) = \frac{48}{5}. \tag{1}$$

We can do a little simplification to see that

$$\left(\frac{4}{52-i}\right)\prod_{j=1}^{i}\left(\frac{49-j}{53-j}\right) = \frac{4}{52\cdot 51\cdot 50\cdot 49}(51-i)(50-i)(49-i),$$

and so the sum above in (1) is equal to

$$\frac{4}{52\cdot 51\cdot 50\cdot 49}\sum_{i=0}^{48} i(51-i)(50-i)(49-i).$$

There is a nicer way to do this problem, however, but it requires more knowledge of probability. We see that there are 4 queens in the deck. Whatever the order of the deck, the queens split the deck into 5 groups -- the cards before the first queen, the cards between the first and second queen, etc. Each of these groups could have from 0 to 48 cards. Let X_1 be the number of cards in the first group, X_2 be the number of cards in the second group, all the way up to X_5. Then we have

$$0 \le X_i \le 48 \text{ and } X_1+X_2+X_3+X_4+X_5 = 48.$$

We know that each X_i is a random variable and because the shuffle is completely random, the distribution of each X_i is the same as all others. Let $\mathbb{E}(Y)$ denote the expected value of the random variable Y. Then we have $\mathbb{E}(X_i) = \mathbb{E}(X_j)$ for all i and j (since they have the same distribution) and thus

$$48 = \mathbb{E}(48) = \mathbb{E}(X_1+X_2+\cdots+X_5) = \mathbb{E}(X_1)+\mathbb{E}(X_2)+\cdots+\mathbb{E}(X_5) = 5\mathbb{E}(X_1)$$

and thus

$$\mathbb{E}(X_1) = \frac{48}{5}.$$

However, this is exactly the number we are looking for (the expected number of cards before the first Queen).