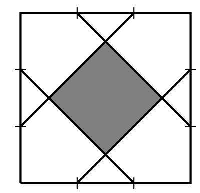
9th Annual Acadia University Undergraduate Mathematics Competition
Time Limit: 2hrs

Instructions: **Full credit will only be given for complete, totally justified solutions**. No calculators or other aids are allowed. Complete your rough work on the scrap paper provided and write your final solutions in the exam booklet. Only the exam booklet needs to be submitted. Have fun!!!

Each of the next three problems are worth 10 points

Problem 1 Consider a square of side-length three. Divide each side of the square into three parts of length one and draw connecting lines as shown in the figure.
What is the area of the shaded square?



Problem 2 A point (x, y) is chosen at random inside a 2×2 square and is used as the center of a circle of radius $1/2$. What is the probability that the circle is completely inside the square?

Problem 3 Let A be a set containing $(2n + 1)$ elements. What is the number of subsets of A which contain at most n elements? Simplify your answer.

Each of the next four problems are worth 15 points

Problem 4 What is the sum of the coefficients in the expanded form of the polynomial $(1 - x^2)^{11}(1 + x^2)^{11}$?

Problem 5 If four squares are chosen at random on a chessboard, find the probability that they all lie on a diagonal line. (Recall that a chessboard is an 8×8 array of squares.)

Problem 6 Which of $(1 + x^2 - x^3)^{1000}$ or $(1 - x^2 + x^3)^{1000}$ has a larger coefficient on x^{20}?

Problem 7 Let t_n be the number of divisors of the positive integer n (where both 1 and n are counted, for example $t_4 = 3$). For $x > 0$, let $[x]$ denote the biggest integer no larger than x (for example $[3.1415] = 3$). Prove that for all positive integers n

$$t_1 + t_2 + t_3 + \cdots + t_n = \left[\frac{n}{1}\right] + \left[\frac{n}{2}\right] + \left[\frac{n}{3}\right] + \cdots + \left[\frac{n}{n}\right].$$

Each of the next three problems are worth 20 points

Problem 8 Suppose that A and B are two $n \times n$ matrices. Prove that

$$(A + AB^{-1}A)^{-1} + (A + B)^{-1} = A^{-1},$$

assuming that all these inverses exist.

Problem 9 Show that for each $n = 1, 2, 3, \ldots$ the series $\displaystyle\sum_{k=1}^{\infty} \frac{k^n}{2^k}$ converges to an even integer.

Problem 10 Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(xy) + f(y - x) \geq f(x + y)$ for all $x, y \in \mathbb{R}$. Prove that $f(x) \geq 0$ for all $x \in \mathbb{R}$.