7th Annual Acadia University Undergraduate Mathematics Competition
Time Limit: 2hrs

Instructions: **Complete credit will only be given for complete, fully justified solutions**. No calculators or other aids are allowed. Complete your rough work on the scrap paper provided and write your final solutions in the exam booklet. Only the exam booklet needs to be submitted. Have fun!!!

The next three problems are each worth 10 points

Problem 1 Find all pairs of integers (positive, negative or zero) x, y so that $x^4 + y^2 = 71$.

Problem 2 The numbers from 1 to 2014 are listed in the following order. First, all numbers not divisible by 3 are listed in increasing order; next, all numbers divisible by 3 but not by3^2 are listed in increasing order; then, all numbers divisible by 3^2 but not 3^3 are listed in increasing order and so on. In which position is 405 (corresponding to April 5th)?

Problem 3 Find all lines which are tangent to both of the parabolas $y = x^2$ and $y = -x^2 + 4x - 4$.

The next four problems are each worth 15 points

Problem 4 If s_n denotes the sum of the first n natural numbers, find the sum of the infinite series

$$\frac{s_1}{1} + \frac{s_2}{2} + \frac{s_3}{4} + \frac{s_4}{8} + \cdots$$

Problem 5 Determine the area of the set of points (x, y) in the plane that satisfy the two inequalities

$$x^2 + y^2 \leq 2 \quad \text{and} \quad x^4 + x^3y^3 \leq xy + y^4.$$

Problem 6
Suppose that x is a real number which satisfies $x^3 + 3x^2 + 4x + 5 = 0$ and y is a real number which satisfies $y^3 - 3y^2 + 4y - 5 = 0$. Compute the value of $(x + y)^{2014}$.

Problem 7 Suppose that each point in the plane is coloured either red, green or blue. Show that there has to be two points with the same colour and a distance of 1 apart.

The next three problems are each worth 20 points

Problem 8 Find all continuous $f : \mathbb{R} \to \mathbb{R}$ which satisfy $f(x + f(y)) = f(x) + y$.

Problem 9 Let N be a fixed positive integer. We flip a fair coin at most N times, stopping either when we have reached N flips or when a tail immediately follows an odd number of heads. Compute the expected number of flips of the coin.

Problem 10 Two parallel lines ℓ_1 and ℓ_2 lie on a plane, a distance of d apart. On ℓ_1 there are an infinite number of points $A_1, A_2, A_3, \ldots$, in that order, with $A_nA_{n+1} = 2$ for all n. On ℓ_2 there are an infinite number of points $B_1, B_2, B_3, \ldots$, in that order and in the same direction, satisfying $B_nB_{n+1} = 1$ for all n. Given that A_1B_1 is perpendicular to both ℓ_1 and ℓ_2, express the sum $\sum_{n=1}^{\infty} \angle A_iB_iA_{i+1}$ in terms of d