5th Annual Acadia University Undergraduate Mathematics Competition
Time Limit: 2hrs

Instructions: More credit will be given for complete, fully justified solutions. No calculators or other aids are allowed. Complete your rough work on the scrap paper provided and write your final solutions in the exam booklet. Only the exam booklet is to be submitted. Have fun!!!

The next three problems are each worth 10 points

Problem 1
What is the side length of the largest equilateral triangle which can be placed inside of a circle of radius one?

Problem 2
Find the sum of the set of all the five digit numbers that can be formed using the digits 1, 2, 3, 4 and 5 where we do not allow repetitions of the digits.

Problem 3
What is the largest number of regions into which a plane can be divided by using n lines?

The next four problems are each worth 15 points

Problem 4
An infinite sequence $b_0, b_1, b_2, \ldots$ satisfies $b_{m-n} + b_{m+n} = b_{2m} + b_{2n}$ for all m and n with $m \geq n \geq 0$. Prove that the sequence is constant (that is, $b_n = b_m$ for all n, m).

Problem 5
Take a stick of unit length. Choose two points on this stick uniformly at random and break the stick at the selected points, forming three parts. What is the probability that you can make a triangle out of these parts?

Problem 6
Show that for any decreasing set of positive numbers $x_1 > x_2 > x_3 > \cdots > x_n$ we have $\left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \leq \sum_{i=1}^{n} \frac{x_i}{\sqrt{i}}$.

Problem 7
Suppose that $x_1, x_2, \ldots, x_n$ are nonnegative real numbers and $x_1 + x_2 + \cdots + x_n < 1/2$. Show that
$(1 - x_1)(1 - x_2)(1 - x_3) \cdots (1 - x_n) > 1/2$.

The next three problems are each worth 20 points

Problem 8
A deck of cards numbered 1 to n (one card for each number) is arranged in random order and placed on the table. If the card numbered k is on top, remove the kth card counted from the top and place it on top of the pile, not otherwise disturbing the order of the cards. Repeat the process. Prove that the card numbered 1 will eventually come to the top, and determine the maximum number of moves that is required to achieve this.

Problem 9
Consider a town with N people. A person sends two letters to two separate people, each of whom is asked to repeat the procedure. Thus, for each letter received, two letters are sent to separate persons chosen at random (irrespective of what happened in the past). What is the probability that in the first n stages, the person who started the chain letter game will not receive a letter?

Problem 10
Take a regular n-gon $\mathcal{G}$ and decompose it into triangles by using diagonals which do not intersect inside $\mathcal{G}$. Show that the number of triangles you obtain only depends on n and not on how you accomplish the decomposition.