4th Annual Acadia University Undergraduate Mathematics Competition
Time Limit: 2hrs

Instructions: More credit will be given for complete, fully justified solutions. No calculators or other aids are allowed. Complete your rough work on the scrap paper provided and write your final solutions in the exam booklet. Only the exam booklet is to be submitted. Have fun!!!

The next three problems are each worth 10 points

Problem 1
Suppose that a cube is inscribed in a sphere of radius one. What is the volume of the cube?

Problem 2
The numbers a, b, c are the digits of a three digit number and satisfy $49a + 7b + c = 286$. What is the three digit number $(100a + 10b + c)$?

Problem 3
If $A = (0, -10)$ and $B = (2, 0)$, find the point(s) C on the parabola $y = x^2$ which minimizes the area of triangle ABC.

The next four problems are each worth 15 points

Problem 4
Find a formula for the sum

$$-1^2 + 2^2 - 3^2 + 4^2 - 5^2 + \cdots + (-1)^n n^2.$$

You must prove that your formula is correct.

Problem 5
Suppose that a new postman delivers the mail to each of the 6 houses on his route at random. If there is one letter addressed to each house, how many ways are there for him to deliver the letters such that he gets none right?

Problem 6
Take an $n > 1$. Show that there is some positive integer N which is a multiple of n and so that all the decimal digits of N are either 0 or 1.

Problem 7
Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy $(x - y)\, f(x + y) - (x + y)\, f(x - y) = 4xy(x^2 - y^2)$.

The next three problems are each worth 20 points

Problem 8
Suppose that a 6×6 square grid of squares is tiled by 1×2 rectangles (dominoes). Show that you can decompose the square grid into two rectangles, each of which is tiled by a disjoint subset of the dominoes.

Problem 9
Let A be a 3×3 matrix with real entries. Suppose that the sum of any row, column or of either of the two diagonals are all equal. Show that the same is true for A^{-1} (but perhaps with a different constant than for A).

Problem 10
Let $\vec{u} \in \mathbb{R}^3$ have unit length. Define the function P by $P(\vec{x}) = \vec{u} \times \vec{x}$ for any $\vec{x} \in \mathbb{R}^3$. Notice that $P : \mathbb{R}^3 \to \mathbb{R}^3$ is linear.

1. Describe the action of the linear function $I + P^2$.
2. Describe the action of the linear function $I + (\sin(\theta))P + (1 - \cos(\theta))P^2$.