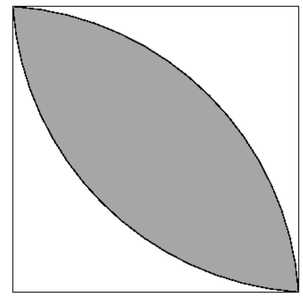
3rd Annual Acadia University Undergraduate Mathematics Competition
Time Limit: 2hrs

Instructions: More credit will be given for complete, fully justified solutions. No calculators or other aids are allowed. Complete your rough work on the scrap paper provided and write your final solutions in the exam booklet. Only the exam booklet is to be submitted. Have fun!!!

The next three problems are each worth 10 points

1. In the diagram, square ABCD has a side length of 6. Circular arcs of radius 6 are drawn with centers B and D (opposite corners). What is the area between the two circular arcs?



2. Suppose that both a and b are positive integers which divide n and that $ab < n$. Does the greatest common divisor of n/a and n/b have to be larger than one?

3. Count the number of ordered quadruples (x_1, x_2, x_3, x_4) of positive odd integers which sum to 44 (that is, so that $x_1 + x_2 + x_3 + x_4 = 44$). Note that in counting them with an order we count $(1, 1, 1, 41)$ as different from $(1, 1, 41, 1)$.

The next four problems are each worth 15 points

4. The graph of $y^2 + 2xy + 40|x| = 400$ breaks the plane up into several regions. What is the area of the bounded region?

5. Let P_n be the probability that after n roles of a seven-sided die the sum of the roles is even. Show that

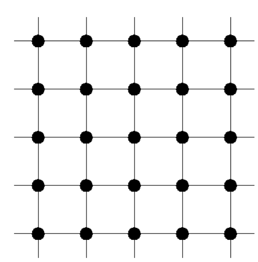
$$\text{Probability of even sum} - \text{Probability of odd sum} = 2P_n - 1 = \frac{(-1)^n}{7^n}.$$

6. Let $a \in \mathbb{R}$ and define the sequence (a_n) by $a_1 = a$ and $a_{n+1} = a_n(1 - a_n)$. Find all values of a so that (a_n) converges.

7. The *2D integer lattice* is the set of all points (x, y) where both x and y are integers. Consider this lattice as a graph, where we connect each point to its four nearest neighbors (up/down and left/right). The task is to colour all the edges and points of this graph with the minimum number of colours. However the colouring must follow all the following rules:

   (a) each pair of adjacent points gets two different colours
   (b) each pair of edges that have a common vertex gets two different colours
   (c) each edge has a different colour than either of its endpoints

   What is the minimum number of colours possible?



The next three problems are each worth 20 points

8. Find all functions $f : \mathbb{R} \to \mathbb{R}$ so that for all $x < y$ we have $f([x, y])$ is a closed interval of length $y - x$.

9. How many $n \times n$ matrices A are there for which all the entries of both A and A^{-1} are either 0 or 1?

10. Evaluate $\displaystyle\sum_{n=2010}^{\infty} \frac{\binom{n}{2010}}{2^n}$.