Compactifications and Function Spaces

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Contents

List of Figures

Summary

In this thesis, we study the partially ordered set of Hausdorff compactifications of a topological space X and its relationship to the space of all bounded continuous real-valued functions on X.

First we prove one kind of generalization of the Hahn-Mazurkeiwicz Theorem. More specifically, we prove there is a perfect surjection between any two locally connected generalized continua with the (α, n) complementation property. We also give some consequences of this to the lattice of compactifications of locally connected generalized continua.

Next we prove that the C^* algebra of all continuous real-valued functions on a compact space X is determined by the lattice structure of the collection of all its closed unital subalgebras. We use this to provide a condition for the isomorphism of the lattice of compactifications of X and the lattice of compactifications of Y. This provides a new proof of a result of Magill. We also obtain many generalizations.

Finally, we discuss the relationships between the lattice of topologies, the lattice of function algebras, and the lattice of partitions in the special case of a finite set. We also comment on when these relations are not the same for infinite sets and give examples of this.

Chapter 1

Introduction

This thesis studies the compactifications of a Tychonoff space X and the relation between the compactifications on X and the algebraic structure of $C^*(X)$, the ring of all bounded continuous real-valued functions on X.

Compact spaces are very nice from a topological point of view. They are "finite" in some topological sense. A compactification of a non-compact space X is a dense embedding of X into some compact space αX. Thus, studying the compactifications of X is the same as studying all the ways that X can be densely embedded in some compact space.

There are many reasons to study compactifications of a space X. One reason is that it is often conceptually simpler to have X as a subspace of a compact space, thus letting you use all of the tools available in the compact setting (boundedness of continuous functions, existence of limit points, *etc*). For instance, compactness is used in many existence proofs by constructing a sequence and then showing that the limit points satisfy the requirements. Since the limit points are in the compactification but not in X, it is important to know what kind of points you are adding to X.

We can also use compactifications to study how "pathological" a bounded continuous function $f : X \to \mathbb{R}$ is. If f extends to the one-point compactification of X (the compactification in which you add only one point to X; for example, S^2 is the one-point compactification of $\mathbb{R}^2$), then $\lim_{x\to\infty} f(x)$ must exist. This means that f is relatively well-behaved. However, clearly not all functions are this well-behaved. For example, let $X = (0, 1)$ and $f(x) = \sin(1/x)$. Then f cannot be extended to the one-point compactification of X. This agrees with our idea of $f(x)$ as being a badly-behaved function as $x \to 0$. So, one use of compactifications is as a measure of how "nice" a function is. Suppose $f : X \to \mathbb{R}$ extends

to a continuous function $f^\alpha : \alpha X \to I\!R$, where αX is a compactification of X. If $r \in I\!R$ is such that $f^{-1}(r)$ is infinite and not compact, then there is some $x \in \alpha X \setminus X$ so that $x \in cl(f^{-1}(r))$. This shows that whether a function f extends to a compactification αX depends on whether there are enough "points" in $\alpha X \setminus X$ to capture the "behavior" of f at "∞". This is very close to the previous idea of studying what kind of limit points you add to a space when you compactify the space.

Another reason for studying compactifications is that sometimes you can extend some structure on X to a corresponding structure on a compactification of X. Consider the natural numbers $I\!N$. For any given space, there is always a "largest" compactification, the *Stone-Čech Compactification* (for the integers this is denoted by $\beta I\!N$). You can extend the addition on $I\!N$ to an addition on $\beta I\!N$ (you do not get a group, but you do get a semi-group). Using this idea and some other related ideas, Neil Hindman was able to use topological techniques to prove some deep combinatorial facts [HI1, HI2, BA] (For example, he proved that if $P_1 \cup P_2 \cup \ldots \cup P_n = I\!N$ then one P_i contains a sequence $\{a_l\}_1^\infty$ so that all finite sums of distinct a_l's are contained in P_i).

The study of the relationship between a topological space X and some kind of algebraic structure defined on X is very common in mathematics. This blending of topology and algebra often yields beautiful structures. Modern mathematics is full of examples of this: homology theory (in all its various incarnations), cohomology theory, sheaf theory, homotopy theory, and algebraic geometry (in as much as it studies algebraic functions defined on a variety), for example. More closely related to this thesis is the study of C^∞ functions on a manifold or the study of bounded measurable functions on a measure space (Ω, Σ, μ). For example, one way to define the tangent bundle of the manifold M (a geometric or analytic object) is that it is the set of all derivations on the algebra $C^\infty(M)$. Similarly, the cotangent bundle can be identified with the quotient space $C^\infty(M)/(C^\infty(M))^2$.

The connection between X and $C^*(X)$ is an old and much-studied subject (Recall that $C^*(X)$ is the set of bounded continous real-valued functions on X). A simple example of the relation between the algebraic structure of $C^*(X)$ and the topology of X is the existence

of non-trivial idempotents in $C^*(X)$. For suppose $e(x) \in C^*(X)$ is such that $e^2 = e$ (*i.e.*, $e(x) \cdot e(x) = e(x)$ for all x in X). Then $e(x) = 0$ or 1 for all $x \in X$. Clearly $e^{-1}(1)$ is a closed and open (clopen) subset of X. Furthermore, $X = e^{-1}(0) \cup e^{-1}(1)$ and this is a disconnection of X. Thus non-trivial idempotents in $C^*(X)$ correspond to clopen subsets of X and the more non-trivial idempotents exist in $C^*(X)$, the more disconnected X is.

If $X = [0,1] \cup [2,3]$, then there are only two non-trivial idempotents and these are $\chi_{[0,1]}$ and $\chi_{[2,3]}$, the characteristic functions of $[0,1]$ and $[2,3]$ respectively. Both of these are primitive idempotents, which means that they cannot be decomposed as an orthogonal sum of other idempotents. An algebraic characterization of a primitive idempotent is that e is a primitive idempotent if there are no non-trivial idempotents e_1 and e_2 so that $e = e_1 + e_2$ and $e_1 \cdot e_2 = 0 = e_2 \cdot e_1$ (this would mean that e_1 and e_2 are orthogonal). Topologically, this translates into requiring that the support of e (the set $e^{-1}(1)$) be connected.

An extreme example is the Cantor set. If X is the Cantor set, then there are no primitive idempotents in $C^*(X)$ – every idempotent can be decomposed into orthogonal pieces. In fact, if X is compact and $C^*(X)$ contains a closed subring $\mathcal{R}$ with no primitive idempotents, then there is a continuous surjection $f : X \to C$ where C is the Cantor set.

However useful idempotents are, they do not tell the whole story. Any connected space has no non-trivial idempotents. The idempotents just give information about the component structure of X. Thus, we must look deeper for connections between the algebraic structure of $C^*(X)$ and the topology of X.

One of the major tools in relating the algebra $C^*(X)$ to the space X is the well-known result that for a compact Hausdorff space X, the maximal ideals in $C^*(X)$ are in one-to-one correspondence with the points of X. We recall that the correspondence is as follows:

$$\text{If } x \in X, \text{ then } M_x \equiv \{f \in C^*(X) | f(x) = 0\} \text{ is a closed ideal.}$$

It is a maximal ideal since $C^*(X)/M_x \cong \mathbb{R}$ (the isomorphism here is just $f \mapsto f(x)$). The fact that this correspondence is surjective is one of the deep and amazing results in this area.

We put a topology on the collection of maximal ideals (see the background chapter for

the details) by using the order structure of the closed ideals in $C^*(X)$. With this topology (called the *hull-kernel topology*), the space of maximal ideals becomes a compact Hausdorff space (also called the *structure space*). Thus, the topology of X is encoded into the algebra of $C^*(X)$. The book [GJ] has a wealth of information on this subject.

One can repeat the construction of the hull-kernel topology on the collection of maximal ideals in an arbitrary commutative ring. Again, this will give you a topological space. However, in general it will not be compact or Hausdorff. However, it is still possible to get algebraic information from the topology of the structure space. M. Henriksen (among many others) has had great success at doing this (see for example [DHKV, H, HJ, HK]).

Another way to look at the correspondence between maximal ideals in $C^*(X)$ and points of X is to use the theory of C^*-algebras. A maximal ideal in $C^*(X)$ corresponds to a multiplicative linear functional on $C^*(X)$ (The functional is the map $C^*(X)/M \to \mathbb{R}$). Thus, we can view the maximal ideals as being contained in the unit ball of the Banach space dual to $C^*(X)$. By Alaoglou's Theorem, this is w^*-compact; and by the Hahn-Banach Theorem, it is Hausdorff.

As discussed in the background chapter, there is a simple and direct correspondence between compactifications of X and closed unital subalgebras of $C^*(X)$. The connection with C^*-algebras makes this relationship clear. Thus, if you want to understand the structure of the collection of compactifications of X, you can use information about the structure of $C^*(X)$. Conversely, if you need information about $C^*(X)$, you can sometimes use information about the compactifications of X to get it.

An example of this is the existence of Banach limits in $l^\infty(\mathbb{N})$. A Banach limit is a continuous linear functional $\phi : l^\infty(\mathbb{N}) \to \mathbb{R}$ so that

$$\inf\{f(n) \mid n \in \mathbb{N}\} \leq \phi(f) \leq \sup\{f(n) \mid n \in \mathbb{N}\}$$

and if $\lim_{n\to\infty}(f(n) - g(n)) = 0$, then $\phi(f) = \phi(g)$. We can view $l^\infty(\mathbb{N})$ as $C^*(\mathbb{N})$ if we give $\mathbb{N}$ the discrete topology. With this identification, to each Banach limit there is some positive Borel probability measure μ supported on $\beta\mathbb{N} \setminus \mathbb{N}$ so that

$$\phi(f) = \int_{\beta\mathbb{N}} f(x)d\mu.$$

This is like evaluating the function f at "∞", except there is no unique choice of "∞". This generalizes the concept of a limit of a sequence by giving all bounded sequences a "limit".

One of the main results in Chapter 4 is the following:

Theorem *Let X and Y be compact Hausdorff spaces. Suppose that $CA(X)$ is lattice isomorphic to $CA(Y)$. Then $C^*(X)$ is isomorphic to $C^*(Y)$.*

Here $CA(X)$ is the partially ordered set of all closed unital subalgebras of $C^*(X)$.

As an example, this theorem says that the collection of all closed unital subalgebras of $l^\infty(\mathbb{N})$ containing c_0 is (partial order) isomorphic to the collection of all compactifications of $\mathbb{N}$.

Organization

The thesis is organized as follows. The first chapter is a collection of standard definitions and results in the theory of compactifications, and is intended to be a review of all the relevant material needed in the later chapters. This material was included in an effort to make the thesis as self-contained as possible. For a more complete account of the basic theory of compactifications see one of the books [Ch, GJ, PW, WA].

The second chapter discusses countable compactifications. A countable compactification is a compactification in which you add a countable number of points to the original space. We specialize to a well-behaved class of spaces – the locally connected generalized continua. We prove a necessary and sufficient condition for a locally connected generalized continua to have a countable compactification of a certain type. We then prove a result (Theorem 36 in Chapter 3) on mapping properties of these locally connected generalized continua. In particular, we prove that if X and Y are locally connected generalized continua with maximal countable compactifications of type (α, n) then there is a perfect surjection $f : X \to Y$. This can be interpreted as one form of a generalization of the Hahn-Mazurkeiwicz Theorem. For reference, here is the statement of the Hahn-Mazurkeiwicz Theorem from [W],

Theorem (Hahn-Mazurkeiwicz) *A Hausdorff space X is a continuous image of* $[0, 1]$ *if and*

only if X is a compact, connected, locally connected metric space.

Using this theorem, one can get a continuous surjection between any two compact, connected, locally connected metric spaces. However, you have little or no control over the images of individual points. Let αX and αY be the maximal countable compactifications of type (α, n) of X and Y respectively. Our generalization gives a surjection $f^\alpha : \alpha X \to \alpha Y$ with the property that $f^{-1}(\alpha Y \setminus Y) = \alpha X \setminus X$. Thus we know where the remainder is mapped under the function.

There is another way to see our Theorem 36 in Chapter 3 as a generalization of the Hahn-Mazurkeiwicz Theorem. The Hahn-Mazurkeiwicz Theorem gives you a perfect surjection between two locally connected generalized continua with the 0-complementation property (*i.e.,* the maximal countable compactification adds 0 points). Our Theorem 36 generalizes this to the (α, n) complementation property for all countable ordinals α and all n.

The third chapter discusses the lattice of Hausdorff compactifications of a space X. Denoting this lattice by $K(X)$, we prove a necessary and sufficient condition for when $K(X)$ is lattice isomorphic to $K(Y)$. This condition is function algebraic in nature. This theorem is seen to be equivalent to a theorem of K. D. Magill from [MG2]. We then prove some extensions of this theorem in the direction as to when a sublattice of $K(X)$ can be isomorphic to a sublattice of $K(Y)$. The main new feature of all these results is the use of function theoretic tools in the proofs of the results. Using these techniques, once we establish the basic framework, we get almost trivial proofs of the extensions.

The final chapter discusses the relationship between the collection of closed algebras of continuous functions on a space, the collection of completely regular topologies on the space, and the collection of partitions of the space. We specialize to the case where the space is a finite set and compare the results in this case with the case in which the space is a compact Hausdorff space. One side result of this analysis is a classification and enumeration of the completely regular topologies on a finite set. This study is a natural specialization of the study of the lattice of compactifications, $K(X)$, in the third chapter.

Chapter 2

Background

In this thesis, all spaces will be completely regular and Hausdorff (unless otherwise stated). We will be concerned with *compactifications* of a space X.

The basic references for this chapter are [Ch] and [GJ]. Most everything in this chapter can be found in one of these books. We gather together here the basic facts which we will need in the subsequent chapters. This has the effect of making this thesis reasonably self-contained. This chapter is a lightning course in the basic results in the theory of compactifications.

Definition 1 *Let X be a non-compact space. A* compactification of X *is a compact space αX and a function $\alpha : X \to \alpha X$ such that α is a dense embedding of X into αX.*

By an abuse of notation, we usually omit the function α and speak of the space αX as a compactification of X, leaving the map α as understood. We also identify X with $\alpha(X) \subset \alpha X$.

Example Let $X = (0,1)$ with the topology it inherits as a subspace of $\mathbb{R}$. Let $\omega X = \mathbf{S}^1$, the unit circle as a subset of the complex plane. Define $\omega : X \to \mathbf{S}^1$ by $\omega(x) = e^{2\pi x i}$. Then clearly ω is an embedding of X into $\mathbf{S}^1$ with $\omega(X)$ dense in $\mathbf{S}^1$. Thus $\mathbf{S}^1$ is a compactification of $(0,1)$. Since $\mathbf{S}^1 \setminus \omega(X) = \{1\}$, a singleton, we say that $\mathbf{S}^1$ is a *one-point compactification* of $(0,1)$.

Again, let $X = (0,1)$. Let $\alpha X = [0,1]$ with its usual topology as a subspace of $\mathbb{R}$. Define $\alpha : X \to [0,1]$ by $\alpha(x) = x$. Clearly this defines an embedding of X into αX and $\alpha(X) =$

(0,1) is dense in [0,1]. Thus, [0,1] is a compactification of (0,1). Since [0,1] \ (0,1) = {0,1}, we say that [0,1] is a *two-point compactification* of (0,1).

We remark that it is is possible to prove that there can only be one one-point compactification of $I\!R$ and one two-point compactification of $I\!R$, up to equivalence. Thus, we will usually say *the* one-point compactification and *the* two-point compactification.

2.1 Equivalence of Compactifications

When do we want to consider two compactifications of X to be the same? We present an example of two compactifications αX and γX of a space X which are homeomorphic but the embeddings of X into αX and γX, respectively, are different. We want to consider these as non-equivalent compactifications. We define an order on the compactifications of X and use this to define when two compactifications are equivalent.

Let αX and γX be two compactifications of X. Then we say that $\alpha X \leq \gamma X$ if there is some continuous surjection $f : \gamma X \to \alpha X$ such that $f|_X = id_X$. This is the same thing as saying that the following diagram commutes:

$$\begin{tikzcd} \gamma X \arrow[rr, "f"] & & \alpha X \\ & X \arrow[ul, "\gamma"] \arrow[ur, "\alpha"'] & \end{tikzcd}$$

Figure 1: Definition of order on $K(X)$

We say that αX is *equivalent to* γX, denoted by $\alpha X \sim \gamma X$, if $\alpha X \leq \gamma X$ and $\gamma X \leq \alpha X$. Suppose $\alpha X \sim \gamma X$, then there is some $f : \gamma X \to \alpha X$ and $g : \alpha X \to \gamma X$ with $f|_X = g|_X = id_X$. This means that f and g are inverses (recall that X is dense in both γX and αX) so αX is homeomorphic to γX.

Now here is our example of homeomorphic but non-equivalent compactifications.

Example Let $X = \{(x, 0) \in I\!R^2 | -1 < x < 1\} \bigcup \{(0, y) \in I\!R^2 | 0 \leq y < 1\}$. Let $Y = cl(X) \subset I\!R^2$ and $\alpha X = Y/((-1, 0) \sim (1, 0))$ and $\gamma X = Y/((1, 0) \sim (0, 1))$. Then

both αX and γX are compactifications of X. In fact, both $\alpha X \setminus X$ and $\gamma X \setminus X$ contain two points. Furthermore αX is homeomorphic to γX. Suppose that $\alpha X \sim \gamma X$ and let $f : \gamma X \to \alpha X$ and $g : \alpha X \to \gamma X$ be the surjections from the definition. Consider the sequences $x_n = (1 - \frac{1}{n}, 0)$ and $y_n = (0, 1 - \frac{1}{n})$. Then in γX we have $\lim x_n = \lim y_n$ so $\lim f(x_n) = \lim f(y_n)$. Since $x_n, y_n \in X$ we have $f(x_n) = x_n$ and similarly $f(y_n) = y_n$. Thus $\lim x_n = \lim y_n$ in αX as well. However, this is not true since $\lim x_n = (1, 0)$ and $\lim y_n = (0, 1)$ and these are not equal in αX. What we have shown is stronger than the fact that αX and γX are not equivalent; we have shown that αX and γX are not related in the order.

Another way to think about this is that we have two different dense embeddings of X into the same compact space. The fact that the compactifications are not equivalent reflects the two different embeddings.

The following picture illustrates this example. The "T" shaped figure (with the end-points) is the space Y. The space αX is the figure on the left, where we have indicated the identification by the large dots. Similarly, the figure on the right is γX. The reason they are not equivalent is because different "arms" of the "T" have been connected.

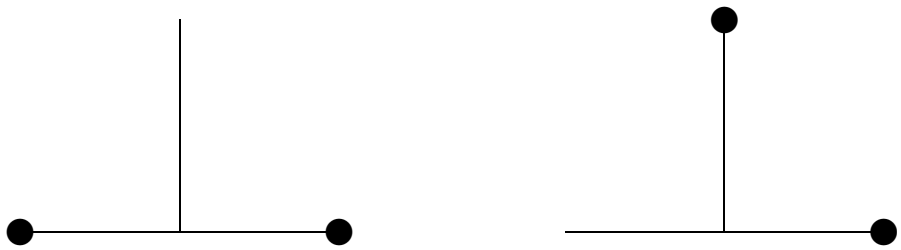

Figure 2: Homeomorphic but non-equivalent compactifications

Not surprisingly, just knowing what spaces are compactifications of X tells us less about X than knowing both the compactifications and the order on the compactifications. This will be a major theme in this thesis.

2.2 Construction of Compactifications

We describe two ways of constructing all the compactifications of a space X. One way is based on embedding the space X into a product of closed intervals and the other way on C^*-algebra theory (or, alternatively, on ring theory via maximal ideals).

First, we give a few definitions.

Definition 2 *Let X be a space. We define*

$$C^*(X) = \{f : X \to I\!R | f \text{ is continuous and bounded } \}$$

and

$$C_0(X) = \{f \in C^*(X) | \forall \epsilon > 0, \exists \text{ compact } K \subset X \text{ with } f(X \setminus K) \subset (-\epsilon, \epsilon)\}.$$

For $f \in C^*(X)$, $|f| = \sup\{|f(x)| \ |x \in X\}$.

Let $\mathcal{F} \subset C^*(X)$. We say that $\mathcal{F}$ separates points from closed sets in X *if for each $x \in X$ and closed set $K \subset X$ with $x \notin K$, there is some $f \in \mathcal{F}$ so that $f(x) \notin cl(f(K))$. We usually say that $\mathcal{F}$* separates points from closed sets *if there is no confusion about the space.*

2.2.1 Compactifications as subsets of products of closed intervals

Let $\mathcal{F} \subset C^*(X)$ separate points from closed sets. For each $f \in \mathcal{F}$ let I_f be a compact interval which contains $f(X)$. Define

$$P = \prod_{f \in \mathcal{F}} I_f.$$

By the Tychonoff Product Theorem, P is compact. We embed X into P by the function $e_{\mathcal{F}} : X \to P$ defined by $e(x)_f = f(x)$. Then clearly $e_{\mathcal{F}}X = cl(e(X)) \subset P$ is a compactification of X.

A natural question then is: Can we get all compactifications of X this way (up to equivalence)?

Definition 3 *Let αX be a compactification of X.*

Define $C_\alpha = \{f \in C^*(X) | f$ extends continuously to $\alpha X\}$.

For each function $f \in C_\alpha$, we denote by f^α the extension of f to αX.

If $\mathcal{F} \subset C_\alpha$ denote by $\mathcal{F}^\alpha$ the set $\{f^\alpha | f \in \mathcal{F}\}$

Note that f^α is unique since X is dense in αX.

Let αX be a compactification of X. We wish to show that the construction above will give a compactification equivalent to αX for some set of functions $\mathcal{F}$. In the construction above let $\mathcal{F} = C_\alpha$.

Proposition 1 *As C^*-algebras, C_α is isomorphic to $C^*(\alpha X)$.*

Proof: The isomorphism is $\psi : C^*(\alpha X) \to C_\alpha$ given by $\psi(f) = f|_X$ for $f \in C^*(\alpha X)$. Clearly, $f|_X \in C_\alpha$ and ψ is an algebra homomorphism. This map is injective since X is dense in αX and it is surjective by the definition of C_α. ∎

This proposition shows that C_α separates points from closed sets in X (since clearly $C^*(\alpha X)$ separates points from closed sets in X).

What we now show is that $e_{\mathcal{F}}X$ is equivalent to αX. Define $\Psi : \alpha X \to P$ by $\Psi(x)_f = f^\alpha(x)$ for all $f \in \mathcal{F}$. We know that Ψ is continuous since the topology on P is the product topology. Notice that for $x \in X$, $e(x)_f = f(x) = \Psi(x)_f$. Thus $\Psi|_X = e_{\mathcal{F}}$ (which is the same as $\Psi|_X = id_X$). This implies that $\Psi(X)$ is dense in $e_{\mathcal{F}}X$. Furthermore, $\Psi(\alpha X)$ is compact so $e_{\mathcal{F}}X \subset \Psi(\alpha X)$. This space X is dense in both αX and $e_{\mathcal{F}}X$ so, in fact, $\Psi(\alpha X) = e_{\mathcal{F}}X$. If $\Psi(x) = \Psi(y)$ then $\Psi(x)_f = \Psi(y)_f$ for all $f \in \mathcal{F}$, or $f^\alpha(x) = f^\alpha(y)$ for all $f \in \mathcal{F}$. However $\mathcal{F}^\alpha$ separates the points of αX (because $\mathcal{F}^\alpha$ is isomorphic to $C^*(\alpha X)$ by Proposition 1), so $x = y$ or Ψ is injective. Thus Ψ is a homeomorphism. This means that $e_{\mathcal{F}}X \leq \alpha X$ since $e_{\mathcal{F}} = \Psi \circ \alpha$. Also, since Ψ is a homeomorphism and $\Psi|_X = id_X$, we have $\Psi^{-1} \circ e_{\mathcal{F}} = \alpha$ so $\alpha X \leq e_{\mathcal{F}}X$ and $e_{\mathcal{F}}X \sim \alpha X$ as desired.

So the answer to the above question is YES – all compactifications of the space X can be obtained this way.

To prove that every compactification of X can be obtained by this construction, we took $\mathcal{F} = C_\alpha$. However, in general, you can use any set of functions which separate points from closed sets. In this more general setting, there is a nice relationship between the set of functions $\mathcal{F}$ and the algebra of functions C_α.

Definition 4 *A* unital algebra $\mathcal{A}$ *is an algebra which contains the multiplicative identity.*

Thus, if $\mathcal{A}$ is an algebra of real-valued functions on X, then $\mathcal{A}$ is unital if and only if $\mathcal{A}$ contains the constant functions.

Proposition 2 *Let $\mathcal{F} \subset C^*(X)$ separate points from closed sets and let αX be the compactification that $\mathcal{F}$ generates. Then $C_\alpha \subset C^*(X)$ is the smallest closed unital algebra which contains $\mathcal{F}$.*

Proof: First we will show that $\mathcal{F} \subset C_\alpha$, *i.e.* that every $f \in \mathcal{F}$ can be extended to $e_{\mathcal{F}}X = \alpha X$. Choose an $f \in \mathcal{F}$. Let $\pi_f : e_{\mathcal{F}}X \to I\!R$ be the projection onto I_f. Then by definition $\pi_f|_X = f$ and π_f is continuous and bounded so $\pi_f = f^\alpha$. Thus, $\mathcal{F} \subset C_\alpha$ as claimed.

By assumption $\mathcal{F}$ separates the points of X. Let $x, y \in e_{\mathcal{F}}X \setminus X$ and suppose that $f^\alpha(x) = f^\alpha(y)$ for all $f \in \mathcal{F}$. Then $\pi_f(x) = \pi_f(y)$ for all $f \in \mathcal{F}$ or $x = y$. Thus $\mathcal{F}^\alpha$ separates the points of αX. By the Stone-Weierstrass Theorem the smallest closed unital algebra which contains $\mathcal{F}^\alpha$ is $C^*(\alpha X)$. Since $C^*(\alpha X) = (C_\alpha)^\alpha$, we are done. ∎

The way to think about this proposition is $\mathcal{F}^\alpha$ is the set of projections onto the coordinates in $e_{\mathcal{F}}X$. The Stone-Weierstrass Theorem then says that $\mathcal{F}^\alpha$ generates all of $C^*(\alpha X)$.

2.2.2 Compactifications as maximal ideal spaces

Let $\mathcal{A}$ be a closed unital subalgebra of $C^*(X)$. Then the set $\hat{\mathcal{A}}$ of all continuous multiplicative linear functionals on $\mathcal{A}$ is a subset of the unit ball of the dual space of $\mathcal{A}$. Furthermore, $\hat{\mathcal{A}}$ is w^*-closed. Thus by Alaoglou's Theorem ([KR], 1.6.5), $\hat{\mathcal{A}}$ is w^*-compact. We embed X

into $\hat{\mathcal{A}}$ by $x \mapsto \hat{x}$ where $\hat{x}(f) = f(x)$ for every $f \in \mathcal{A}$. To show that $\hat{x} \in \hat{\mathcal{A}}$, let $f, g \in \mathcal{A}$. Then $\hat{x}(fg) = f(x)g(x) = \hat{x}(f)\hat{x}(g)$ and $\hat{x}(f+g) = (f+g)(x) = f(x)+g(x) = \hat{x}(f)+\hat{x}(g)$. Similarly, if $\lambda \in I\!R$, then $\hat{x}(\lambda f) = (\lambda f)(x) = \lambda f(x) = \lambda \hat{x}(f)$. Thus $\hat{x} \in \hat{\mathcal{A}}$. This mapping is continuous since if $\{x_\gamma\} \subset X$ is a net with $x_\gamma \to z$, then $f(x_\gamma) \to f(z)$ for all $f \in \mathcal{A}$ so $\hat{x_\gamma} \to \hat{z}$ in $\hat{\mathcal{A}}$.

Now we show that the mapping $x \to \hat{x}$ embeds X densely in $\hat{\mathcal{A}}$. Suppose not, then there is a non-empty open set $U \subset \hat{\mathcal{A}} \setminus X$. However, $\hat{\mathcal{A}}$ is compact and Hausdorff (compact by Alaoglou's Theorem and Hausdorff by the Hahn-Banach Theorem) hence normal; thus by Urysohn's Lemma, there is an $f \in C^*(\hat{\mathcal{A}})$ so that $f \neq 0$ and $f(x) = 0$ for all $x \in X$. However, by the Gelfand Representation Theorem ([KR],4.4.3), $C^*(X)$ and $C^*(\hat{\mathcal{A}})$ are isomorphic so, if $f(x) = 0$ for all $x \in X$, then $f = 0$. This is a contradiction. Thus, no such non-empty set U can exist, or X is densely embedded in $\hat{\mathcal{A}}$ and $\hat{\mathcal{A}}$ is a compactification of X.

What does this have to do with maximal ideals? Well, if ϕ is a multiplicative linear functional on $\mathcal{A}$, then $\phi^{-1}(0)$ is a maximal ideal of $\mathcal{A}$. Conversely, if M is a maximal ideal in $\mathcal{A}$, then the natural quotient map $q : \mathcal{A} \to \mathcal{A}/M \cong I\!R$ is a multiplicative linear functional. There is thus a natural correspondence between the set of all multiplicative linear functionals on an algebra and the set of maximal ideals of the algebra. Furthermore, one can topologize the maximal ideals by carrying the w^*-topology from $\hat{\mathcal{A}}$. A direct way of describing this topology on the set of all maximal ideals is the following: For S a collection of maximal ideals, the closure of S (here denoted by $\overline{S}$) is the set

$$\overline{S} = \{M | M \text{ maximal ideal }, M \supseteq \bigcap \{I | I \in S\}\}.$$

In the language of multiplicative linear functionals, this is

$$\overline{S} = \{\phi | \phi^{-1}(0) \supseteq \bigcap \{\psi^{-1}(0) | \psi \in S\}\}.$$

We now show that this topology on $\hat{\mathcal{A}}$ (called the *hull-kernel topology*) is the same as the w^*-topology. Let $J = \bigcap \{\psi^{-1}(0) | \psi \in S\}$. Suppose that $\phi_\alpha \in S$ with $\phi_\alpha \to \phi$ in the w^*-topology, *i.e.* that $\phi_\alpha(f) \to \phi(f)$ for all $f \in C^*(X)$. Since $\phi_\alpha \in S$, if $f \in J$, then $\phi_\alpha(f) = 0$ so $\phi(f) = 0$. Thus $\phi \in \overline{S}$ or $\overline{S}$ is w^*-closed. Thus, w^*-cl(S) $\subset \overline{S}$.

We wish to show that $\overline{S} \subset w^*$-cl(S). If $J \neq 0$, then we can consider the algebra $C^*(X)/J$, so without loss of generality $J = 0$. In this case, we want to show that w^*-cl(S) is the set of all continuous multiplicative linear functionals on $C^*(X)$. Define the function

$$\Psi : C^*(X) \to \oplus_{\phi \in S} I\!R$$

by $\Psi(f)_\phi = \phi(f)$. Then it is easy to check that Ψ is an algebra homomorphism. Since $J = 0$, Ψ is injective. Thus $f \geq 0$ if and only if $\Psi(f) \geq 0$ if and only if $\phi(f) \geq 0$ for all $\phi \in S$. Suppose that there is some $\hat{\phi} \notin w^*$-cl(S). Then $\hat{\phi}$ is not an element of the w^*-closed convex hull of S so, by the Hahn-Banach Theorem applied to $(C^*(X))^*$ with the w^*-topology, there is some $g \in C^*(X)$ with $\hat{\phi}(g) > a = \sup\{\phi(g) | \phi \in S\}$. Then $\phi(a - g) = a - \phi(g) \geq 0$ for all $\phi \in S$ so $a - g \geq 0$. However, this contradicts the fact that $\hat{\phi}(a - g) = a - \hat{\phi}(g) < 0$ (since $\hat{\phi}$ preserves order, being a multiplicative linear functional). Thus, no such $\hat{\phi}$ can exist or the w^*-closure of S is the set of all continuous multiplicative linear functionals on $C^*(X)$. (This argument is adapted from arguments in [KR] and [MUR]).

We can use either description of the topology on the maximal ideal space (the space of all continuous multiplicative linear functionals). We will use the one that makes the ideas and proofs clearer.

We warn the reader that the above constructions depend very heavily on the fact that the algebra in question, $C^*(X)$, is commutative. Things are a lot more complicated in the case of a non-commutative algebra.

Again, a natural question is: Can every compactification of X be obtained this way? (and again the answer is yes.)

To this end, let αX be a compactification of X. For $\mathcal{A}$ we take C_α (just as before). Since C_α is isomorphic to $C^*(\alpha X)$ as a C^*-algebra, we know that C_α is a closed subalgebra of $C^*(X)$. We claim that $\hat{\mathcal{A}}$ is homeomorphic to αX and the homeomorphism is actually an equivalence of compactifications. The homeomorphism goes as follows: for $x \in \alpha X$, $\psi(x) = \hat{x}$, where $\hat{x}(f) = f^\alpha(x)$ for $x \in \alpha X$. Clearly ψ is injective and continuous and $\psi|_X = ed_X$. Gelfand's Theorem says that $C^*(\alpha X) \cong C^*(\hat{\mathcal{A}})$ so by Theorem 4.9 from [GJ] we know that αX and $\hat{\mathcal{A}}$ are homeomorphic and so are equivalent.

If we start with some closed unital algebra $\mathcal{A} \subset C^*(X)$ and use the C^*-algebra method to generate a compactification αX, what is the relationship between $\mathcal{A}$ and C_α?

Proposition 3 $\mathcal{A} = C_\alpha$

Proof: Clearly $\mathcal{A} \subset C_\alpha$, since each function in $\mathcal{A}$ can be extended to $\hat{\mathcal{A}} = \alpha X$. Now $C^*(\alpha X)|_X = C_\alpha$, by Proposition 2, and by the Gelfand Representation Theorem $C^*(\alpha X)|_X = \mathcal{A}$. Thus $\mathcal{A} = C_\alpha$ as claimed. ∎

2.2.3 Examples: One-point and two-point compactifications of (0,1)

Let us look at these two examples using both constructions to see what is going on.

Clearly, the one-point compactification of $(0,1)$ is the circle $\mathbf{S}^1$ and the two-point compactification of $(0,1)$ is the interval $[0,1]$.

One-point compactification of (0,1)

Let $\mathcal{A} \subset C^*(0,1)$ be the collection of all $f \in C^*(0,1)$ such that $\lim_{x\to 1} f(x) = \lim_{x\to 0} f(x)$ exists.

First we see how the one-point compactification can be seen as a subspace of a product of compact intervals. We wish to show that $e_{\mathcal{A}}X$ is the one-point compactification of X = (0,1). To this end, define $\psi : \mathbf{S}^1 \to e_{\mathcal{A}}X$ by $\psi(e^{2\pi xi}) = e_{\mathcal{A}}(x)$ for $x \in (0,1)$ and $\psi(1) = \psi(e^0) = z$ where $(z)_f = \lim_{x\to 0} f(x)$. Clearly $\psi|_X = e_{\mathcal{A}}$ and if $x_n \to 0$ then $\psi(e^{2\pi x_n i})_f = f(x_n) \to f(0)$ for all $f \in \mathcal{A}$. Thus ψ is continuous. It is trivial that ψ is injective. Since $\psi(\mathbf{S}^1)$ is compact and $\psi(\mathbf{S}^1) \supset e_{\mathcal{A}}(X)$ we have $\psi(\mathbf{S}^1) \supset cl(e_{\mathcal{A}}(X)) = e_{\mathcal{A}}X$. Therefore, ψ is a homeomorphism.

Let us look at the one-point compactification as a subset of the set of multiplicative linear functionals on the C^*-algebra $\mathcal{A}$. Here we want to show that $\hat{\mathcal{A}}$ is homeomorphic to $\mathbf{S}^1$. To do this, define a function $\phi : \mathbf{S}^1 \to \hat{\mathcal{A}}$ by $\phi(e^{2\pi xi}) = \hat{x}$ if $x \in (0,1)$ and $\phi(1) = \phi(e^0) = \hat{z}$, where $\hat{z}(f) = \lim_{x->1} f(x)$ for all $f \in \mathcal{A}$. We must show that $\hat{z} \in \hat{\mathcal{A}}$. To this end consider

the following:

$$\hat{z}(fg) = \lim_{x\to 1} fg(x) = \lim_{x\to 1} f(x) \lim_{x\to 1} g(x) = \hat{z}(f)\hat{z}(g)$$

$$\hat{z}(f+g) = \lim_{x\to 1} f(x) + g(x) = \lim_{x\to 1} f(x) + \lim_{x\to 1} g(x) = \hat{z}(f) + \hat{z}(g)$$

$$\hat{z}(\lambda f(x)) = \lambda \lim_{x\to 1} f(x) = \lambda \hat{z}(f)$$

Furthermore, $|\hat{z}(f)| = |\lim_{x\to 1} f(x)| \leq \sup_{x\in(0,1)} |f(x)| = |f|$. Thus, $\hat{z} \in \hat{\mathcal{A}}$. The same arguments which showed that ψ is a homeomorphism show that ϕ is a homeomorphism.

Notice that the algebra $\mathcal{A}$ is the set of all bounded continuous functions on (0,1) which will extend to $\mathbf{S}^1$, the one-point compactification of (0,1).

Two-point compactification of (0,1)

Let $\mathcal{B} \subset C^*(0,1)$ such that if $f \in \mathcal{B}$ then $\lim_{x\to 1} f(x)$ exists and $\lim_{x\to 0} f(x)$ exists. We do not require them to be equal. Notice that $\mathcal{A} \subset \mathcal{B}$.

Again, first we want to see the two-point compactification as a subset of a product space. We want to show that $e_{\mathcal{B}}X$ is homeomorphic to [0,1]. Define $\Psi : [0,1] \to e_{\mathcal{B}}X$ by $\Psi(x) = e_{\mathcal{B}}(x)$ for $x \in (0,1)$, $\Psi(0) = a$ where $a_f = \lim_{x\to 0} f(x)$ for all $f \in \mathcal{B}$ and $\Psi(1) = b$ where $b_f = \lim_{x\to 1} f(x)$ for all $f \in \mathcal{B}$. Again, clearly Ψ is continuous and injective and surjectivity follows by the same type of argument as before. Thus, $e_{\mathcal{B}}X$ is homeomorphic to [0,1].

Next we wish to see the two-point compactification of (0,1) via the C^*-algebra construction. To show that $\hat{\mathcal{A}}$ is homeomorphic to [0,1], we define $\Phi : [0,1] \to \hat{\mathcal{A}}$ by $\Phi(x) = \hat{x}$ for $x \in (0,1)$ and $\Phi(0) = \hat{a}$ where $\hat{a}(f) = \lim_{x\to 0} f(x)$ for all $f \in \mathcal{B}$ and $\Phi(1) = \hat{b}$ where $\hat{b}(f) = \lim_{x\to 1} f(x)$ for all $f \in \mathcal{B}$. Just as before, it is easy to check that the map Φ is a homeomorphism.

Again, notice that the algebra $\mathcal{B}$ is the set of all continuous bounded functions on (0,1) which will extend to [0,1].

In both of the examples above, you get the same compactification with both constructions. This is always the case as we show now.

Proposition 4 *Let $\mathcal{A}$ be a closed unital subalgebra of $C^*(X)$. Then using either of the above constructions with $\mathcal{A}$, you get the same compactification of X.*

Proof: Define $\psi : \alpha X \to \hat{\mathcal{A}}$ by $x \mapsto \hat{x}$ where $\hat{x}(f) = f^\alpha(x)$. Since $\mathcal{A} = C_\alpha$ by Proposition 1, this is well defined. Clearly ψ is continuous. Suppose that $\psi(x) = \psi(y)$, then $\hat{x}(f) = \hat{y}(f)$ for all $f \in \mathcal{A}$ or $f^\alpha(x) = f^\alpha(y)$ for all $f \in \mathcal{A}$. By Proposition 1 and Proposition 2 we know $\mathcal{A} = C_\alpha \cong C^*(\alpha X)$, so $\mathcal{A}$ separates the points of αX. This implies that $x = y$, or ψ is injective. Since αX is compact, ψ is a homeomorphism into $\hat{\mathcal{A}}$. Since $\psi(\alpha X) \supset X$ and $cl(X) = \hat{\mathcal{A}}$, then ψ is a homeomorphism onto $\hat{\mathcal{A}}$. ∎

In a very real sense, there is little difference between the two constructions. Both use function spaces, for one thing. However, even more is true. Let $\mathcal{A} \subset C^*(X)$ be a closed unital algebra and $x \in e_{\mathcal{A}}X \equiv \alpha X$. Then x is a function on $\mathcal{A}$ with $x(f) = \pi_f(x) = f^\alpha(x)$, where π_f is the projection onto I_f and f^α is the extension of f to αX. We claim that x is in fact a continuous multiplicative linear functional on $\mathcal{A}$, so $x \in \hat{\mathcal{A}}$. To show this, let $f, g \in \mathcal{A}$, then

$$x(f+g) = \pi_{f+g}(x) = (f+g)^\alpha(x) = f^\alpha(x) + g^\alpha(x) = \pi_f(x) + \pi_g(x) = x(f) + x(g)$$

and

$$x(fg) = \pi_{fg}(x) = (fg)^\alpha(x) = f^\alpha(x)g^\alpha(x) = \pi_f(x)\pi_g(x) = x(f)x(g)$$

and for $\lambda \in I\!R$,

$$x(\lambda f) = \pi_{\lambda f}(x) = (\lambda f)^\alpha(x) = \lambda f^\alpha(x) = \lambda \pi_f(x) = \lambda x(f).$$

Furthermore,

$$|x(f)| = |\pi_f(x)| = |f^\alpha(x)| \leq \sup\{|f(x)| \; |x \in X\} = |f|.$$

So x is indeed a continuous multiplicative linear functional on $\mathcal{A}$ and thus is an element of $\hat{\mathcal{A}}$.

The topology on both $\hat{\mathcal{A}}$ and $e_{\mathcal{A}}X$ is the topology of pointwise convergence, so it is not surprising that they should be the same. Furthermore, the fact that $\hat{\mathcal{A}}$ is compact is

proved using Alaoglou's Theorem, which is a relatively simple consequence of the Tychonoff Product Theorem. Thus even though the two methods seem to be different, they really are not.

2.3 The Stone-Čech Compactification

What happens if we take all of $C^*(X)$ in either of the above constructions? We will get a compactification of X called the *Stone-Čech compactification*, denoted by βX. The Stone-Čech compactification is (perhaps) the most important compactification of a space. In the order on the compactifications, βX is the largest compactification of X. Unlike the examples above (the one-point and two-point compactifications of (0,1)), it is usually difficult or impossible to give a description of βX other than to list its properties. We will now prove some important properties of βX.

Theorem 5 *Each $f \in C^*(X)$ has an extension to βX. In other words, $C_\beta = C^*(X)$.*

Proof: By Proposition 4, we can use either construction to describe βX. Let $f \in C^*(X)$. Then $f^\beta = \pi_f : \beta X \to I\!R$ is the desired extension (this works since we used all of $C^*(X)$ in the construction of βX). ∎

We show that βX is the only compactification with this property (see Theorem 7 below). Using the preceeding theorem, we can prove an even stronger result.

Theorem 6 *Let H be a compact Hausdorff space and $\phi : X \to H$. Then there is a function $\phi^\beta : \beta X \to H$ so that $\phi^\beta|_X = \phi$.*

Proof: Let $e : H \to \prod I_f$ be the embedding of H into a product of intervals, where the product is over all $f \in C^*(H)$. For each $f \in C^*(H)$ we have $f \circ \phi \in C^*(X)$ so there is an extension $(f \circ \phi)^\beta : \beta X \to I\!R$. Define $\psi : \beta X \to \prod I_f$ by $\psi(x)_f = (f \circ \phi)^\beta(x)$. Clearly ψ is continuous. Since $\psi(\beta X) \subset e(H)$ and e is an embedding of H into $\prod I_f$, we can define $\phi^\beta : \beta X \to H$ by $\phi^\beta(x) = e^{-1}(\psi(x))$. If $x \in X$ then $\psi(x)_f = (f \circ \phi)^\beta(x) =$

$(f \circ \phi)(x) = f(\phi(x)) = e(\phi(x))_f$ so $\phi|_X = e \circ \phi$. This implies that $\phi^\beta|_X = \phi$ (the diagram below illustrates this). ∎

$$\begin{array}{ccc} \beta X & \stackrel{\phi^\beta}{\rightarrow} & \prod I_f \\ \uparrow & & \uparrow e \\ X & \stackrel{\phi}{\rightarrow} & H \end{array}$$

Figure 3: Extension of $f : X \to H$ to $f^\beta : \beta X \to H$

This last result yields another nice fact. Let $\phi : X \to Y$ where X and Y are spaces. Then there is a $\phi^\beta : \beta X \to \beta Y$ so that $(\phi^\beta)|_X = \phi$. This is because $\phi : X \to Y \subset \beta Y$ so ϕ is a function from X into a compact Hausdorff space βY and, hence, has an extension $\phi^\beta : \beta X \to \beta Y$.

2.4 K(X) – the Lattice of Compactifications

Once we are a bit careful about definitions (to avoid paradoxes like the set of all sets), the collection of all compactifications of X forms a complete upper semi-lattice (a complete upper semi-lattice is a partially ordered set which contains all suprema of collections). We will discuss these definitions and some properties of this semi-lattice.

First, we prove a useful characterization of βX, up to equivalence.

Theorem 7 *Let γX be a compactification of X. Then every $f \in C^*(X)$ extends to γX if and only if $\gamma X \sim \beta X$.*

Proof: Suppose that every $f \in C^*(X)$ extends to γX. We want to show that $\gamma X \sim \beta X$. It suffices to prove that $\beta X \leq \gamma X$. We use the same idea as in the proof of Theorem 6 to do this. Define $\phi : \gamma X \to \prod I_f$ by $\phi(x) = f^\gamma(x)$, where the product is taken over all $f \in C^*(X)$ and f^γ is the extension of f to γX. Since X is dense in γX and in βX and since γX is compact, $\phi : \gamma X \to \beta X$ is surjective. Clearly $\phi|_X = id_X$.

Conversely, suppose that $\gamma X \sim \beta X$ by $\phi : \gamma X \to \beta X$. Let $f \in C^*(X)$. Then $f^\gamma = (f^\beta \circ \phi) : \gamma X \to I\!R$ is continuous and satisfies $f^\gamma|_X = f$. ∎

This is a very nice result. It says that βX is the only compactification (up to equivalence) of X to which every bounded real-valued function extends.

We now define $K(X)$, the collection of compactifications of X. Instead of taking the collection of all compactifications of X, we take just one representative from each equivalence class of compactifications. The following definition makes this precise.

Definition 5 *Let βX be the Stone-Čech compactification of X. Let C be a partition of βX and $q : \beta X \to C$ be the natural quotient map. Endow C with the quotient topology. We define $K(X)$ to be the set of all such C so that C is Hausdorff and $q|_X = id_X$.*

First we show that every $C \in K(X)$ is a compactification of X. Let $C \in K(X)$ and $q : \beta X \to C$ be the map from the definition. Since $q|_X = id_X$, we can think of $X \subset C$. Also $cl(X) = C$ since X is dense in βX and $C = q(cl(X)) = q(\beta X) \subset cl(q(X))$ because q is continuous. Therefore, C is a compactification of X.

Next, we show that every equivalence class of compactifications is represented in $K(X)$. Let αX be a compactification of X. Then $\alpha : X \to \alpha X$ so by Theorem 6 there is some function $f : \beta X \to \alpha X$ with $f|_X = \alpha = id_X$. Since α embeds X densely in αX, the function f is surjective. There is no reason that αX should be a partition of βX. However, the map $f : \beta X \to \alpha X$ induces the partition $\mathcal{P} = \{f^{-1}(x) | x \in \alpha X\}$ of βX. Each $S \in \mathcal{P}$ is identified with a unique point of αX so we can topologize $\mathcal{P}$ in such a way as to make it homeomorphic to αX. Clearly then $\mathcal{P} \in K(X)$ and $\mathcal{P} \sim \alpha X$. Thus, every equivalence class of compactifications is represented in $K(X)$.

The last thing necessary is to show that no equivalence class has two representatives in $K(X)$. In order to do this, it is easier to prove a proposition first.

Consider the identity map $id_X : X \to X \subset \alpha X$ where αX is a compactification of X. This map has an extension $\pi_\alpha^\beta : \beta X \to \alpha X$ which is a surjection. It is completely determined by αX. We call this map the *canonical projection* of βX onto αX.

Definition 6 *Let αX be a compactification of X and let $\pi_\alpha^\beta : \beta X \to \alpha X$ be the canonical projection. The β-*family of αX *is the set $F_\alpha = \{(\pi_\alpha^\beta)^{-1}(x) | x \in \alpha X\}$.*

Notice that F_α is a partition of βX and that for each $x \in X \subset \alpha X$ we have $(\pi_\alpha^\beta)^{-1}(x) = \{x\}$, a singleton.

Proposition 8 *Let $\alpha X, \gamma X \in K(X)$. Then $\alpha X \leq \gamma X$ if and only if F_γ refines F_α.*

Proof: Let $\pi_\alpha^\beta : \beta X \to \alpha X$ and $\pi_\gamma^\beta : \beta X \to \gamma X$ be the canonical projections.

Suppose that $\alpha X \leq \gamma X$, given by $\pi_\alpha^\gamma : \gamma X \to \alpha X$. Then $\pi_\alpha^\beta|_X = id_X = (\pi_\alpha^\gamma \circ \pi_\gamma^\beta)|_X$ so $\pi_\alpha^\beta = \pi_\alpha^\gamma \circ \pi_\gamma^\beta$, since X is dense in βX. Let $S \in F_\alpha$, then $S = (\pi_\alpha^\beta)^{-1}(x)$ for some $x \in \alpha X$. However, then also $S = (\pi_\alpha^\gamma \circ \pi_\gamma^\beta)^{-1}(x) = (\pi_\gamma^\beta)^{-1}((\pi_\alpha^\gamma)^{-1}(x))$ which means that $(\pi_\gamma^\beta)^{-1}(y) \subset S$ for all $y \in (\pi_\alpha^\gamma)^{-1}(x)$. Therefore F_γ refines F_α.

Conversely, suppose that F_γ refines F_α. We need to construct $\pi_\alpha^\gamma : \gamma X \to \alpha X$ so that $\pi_\alpha^\gamma|_X = id_X$. Define π_α^γ by $\pi_\alpha^\gamma(x) = y$ where $(\pi_\gamma^\beta)^{-1}(x) \subset (\pi_\alpha^\beta)^{-1}(y)$. This is well-defined since F_γ refines F_α. Furthermore $\pi_\alpha^\gamma|_X = id_X$ since $(\pi_\alpha^\beta)^{-1}(x) = (\pi_\gamma^\beta)^{-1}(x) = \{x\}$ for $x \in X$. The function π_α^γ is continuous since αX and γX have the quotient topology by π_α^β and π_γ^β respectively and $\pi_\alpha^\beta = \pi_\alpha^\gamma \circ \pi_\gamma^\beta$. ∎

Using this proposition we now show that each equivalence class has only one representative in $K(X)$. Suppose that αX and γX are in $K(X)$ with $\alpha X \sim \gamma X$. Then by Proposition 8 F_α refines F_γ and F_γ refines F_α, so that $F_\alpha = F_\gamma$. However, $\alpha X \in K(X)$ implies that $F_\alpha = \alpha X$ and similarly $\gamma X = F_\gamma$. Thus $\alpha X = \gamma X$.

Notice: For the rest of this thesis, we will not distinguish between equivalent compactifications.

The definition of $K(X)$ makes it clear that βX is the maximum element in $K(X)$, so $\bigvee\{\gamma X | \gamma X \in K(X)\}$ always exists. In fact, $K(X)$ is always a complete upper semi-lattice.

Proposition 9 *$K(X)$ is a complete upper semi-lattice.*

Proof: Let $\{\alpha_i X\} \subset K(X)$ with $\alpha_i : X \to \alpha_i X$ the embeddings. We want to show that $\bigvee\{\alpha_i X\}$ exists. To this end, let $P = \prod_i \alpha_i X$ and notice that P is compact. Define $e : X \to P$ by $e(x)_i = \alpha_i(x)$ and let $\alpha X = cl(e(X)) \subset P$. Clearly αX is a compactification of X. We claim that $\alpha X \geq \alpha_i X$ for all i. To see this, define $\pi^{\alpha}_{\alpha_i} : \alpha X \to \alpha_i X$ by projection onto the ith coordinate. Since X is dense in both αX and $\alpha_i X$ and since αX is compact, $\pi^{\alpha}_{\alpha_i}$ is surjective. Thus $\alpha X \geq \alpha_i X$.

Now suppose that $\gamma X \geq \alpha_i X$ for all i and let $\pi^{\gamma}_i : \gamma X \to \alpha_i X$ be the projections which show this. Define $q : \gamma X \to P$ by $(q(x))_i = \pi^{\gamma}_i(x)$. Since X is dense in all of γX, αX, and $\alpha_i X$ and all of these are compact, $q : \gamma X \to \alpha X$ is surjective and $f|_X = id_X$. Therefore, $\gamma X \geq \alpha X$ or αX is the least upper bound of $\{\alpha_i X\}$. ∎

If X is locally compact, then X has a one-point compactification (in fact, the existence of a one-point compactification is equivalent to X being locally compact). Suppose X is locally compact and let ωX be the one-point compactification of X. Then if γX is any other compactification of X we have $\omega X \leq \gamma X$. We prove these statements now.

Proposition 10 *The space X is locally compact if and only if it has a one-point compactification.*

Proof: Suppose that ωX is a one-point compactification of X. Then X is open in ωX, since $\omega X \setminus X$ is closed. An open subset of a locally compact space is locally compact so X is locally compact.

Conversely, suppose that X is locally compact. Let $\omega X = X \bigcup \{\infty\}$ where $\infty \notin X$. We topologize ωX as follows: for each $x \in X$, we give x the neighborhood base it has from X and for ∞, we let the collection $\{\omega X \setminus K | K \subset X,$ and K is compact $\}$ be a neighborhood base. Then it is easy to show that ωX with this topology is a compactification of X. ∎

There is another way to prove the existence of ωX for locally compact X. Recall that $C_0(X)$ is the set of all functions on X which "vanish at infinity". Since X is locally compact,

$C_0(X)$ separates points from closed sets. Now let $\mathcal{A}$ be the collection of all $f \in C^*(X)$ with $\lim_{x\to\infty} f(x)$ exists. Then $C_0(X) \subset \mathcal{A}$, so $\mathcal{A}$ also separates poins from closed sets. Furthermore, using either construction we get ωX, the one-point compactification of X.

To show this, we need only show that there is only one element of $\hat{\mathcal{A}} \setminus X$. Recall that $\hat{\mathcal{A}}$ is the collection of continuous multiplicative linear functionals on $\mathcal{A}$. Clearly, $\lim_{x\to\infty} f(x)$ exists for all $f \in \mathcal{A}$ (by the definition of $\mathcal{A}$). Thus $\zeta(f) = \lim_{x\to\infty} f(x)$ is an element of $\hat{\mathcal{A}}$. Let $\hat{z} \in \hat{\mathcal{A}} \setminus X$, then there is some net $\hat{x}_\gamma \subset \hat{X}$ so that $\hat{x}_\gamma \to \hat{z}$ in the w^*-topology (*i.e.* $f(x_\gamma) \to f(z)$ for all $f \in \mathcal{A}$). However, $\hat{z} \in \hat{\mathcal{A}} \setminus X$ implies that the net x_γ has no cluster points in X so for all compact $K \subset X$ there is some δ so that if $\gamma \geq \delta$ then $x_\gamma \notin K$. This implies that $f(x_\gamma) \to \lim_{x\to\infty} f(x) = \zeta(f)$. Thus, $\hat{z} = \zeta$, or $\hat{\mathcal{A}} \setminus X = \{\zeta\}$.

Next, we show that ωX is the minimum in $K(X)$, for any one-point compactification ωX.

Proposition 11 *Let X be locally compact and let ωX be a one-point compactification of X. Then for any $\gamma X \in K(X)$, we have $\omega X \leq \gamma X$.*

Proof: Let ∞ be the unique point in $\omega X \setminus X$. We define $\pi^\gamma_\omega : \gamma X \to \omega X$ by $\pi^\gamma_\omega(x) = x$ when $x \in X$ and $\pi^\gamma_\omega(x) = \infty$ when $x \in \gamma X \setminus X$. Clearly π^γ_ω is surjective and $\pi^\gamma_\omega|_X = id_X$. Furthermore, clearly $\pi^\gamma_\omega|_X$ and $\pi^\gamma_\omega|_{\gamma X\setminus X}$ are both continuous. Let U be a neighborhood of ∞, so that $U = \omega X \setminus K$ with $K \subset X$ compact. Then $(\pi^\gamma_\omega)^{-1}(U) = (\pi^\gamma_\omega)^{-1}(\omega X \setminus K) = \gamma X \setminus (\pi^\gamma_\omega)^{-1}(K)$ is open. Thus π^γ_ω is continuous. ∎

Notice that we have proved that the one-point compactification is unique up to equivalence (and, thus, up to homeomorphism). So we have justified our calling it *the* one-point compactification.

So if X is locally compact, $K(X)$ has both a minimum and a maximum. In fact, in this case $K(X)$ is a complete lattice. However before proving this, we first prove a theorem which gives a nice relation between the C_α's and the order on $K(X)$.

Theorem 12 *Let $\alpha X, \gamma X \in K(X)$. Then $\alpha X \leq \gamma X$ if and only if $C_\alpha \subset C_\gamma$.*

Proof: Suppose that $\alpha X \leq \gamma X$ by $\pi_\alpha^\gamma : \gamma X \to \alpha X$. Let $f \in C_\alpha$, then $f^\gamma \equiv f^\alpha \circ \pi_\alpha^\gamma : \gamma X \to I\!R$ is in C_γ since $f^\gamma|_X = (f^\alpha \circ \pi_\alpha^\gamma)|_X = f^\alpha|_X = f$ (since $\pi_\alpha^\gamma|_X = id_X$). Thus $C_\alpha \subset C_\gamma$.

Conversely, suppose that $C_\alpha \subset C_\gamma$. We wish to show that $\alpha X \leq \gamma X$. We can use either construction to generate αX and γX, so we view $\alpha X = cl(e_\alpha(X)) \subset \prod\{I_f | f \in C_\alpha\}$ and $\gamma X = cl(e_\gamma(X)) \subset \prod\{I_f | f \in C_\gamma\}$. Define $\phi : \prod\{I_f | f \in C_\gamma\} \to \prod\{I_f | f \in C_\alpha\}$ by $\phi(z)_f = z_f$ (this is just projecting out the coordinates in $\prod\{I_f | f \in C_\gamma\}$ which are not in $\prod\{I_f | f \in C_\alpha\}$). Let $\pi_\alpha^\gamma : \gamma X \to \prod\{I_f | f \in C_\alpha\}$ be the restriction of ϕ to γX. Since X is dense in γX and αX and since γX is compact, $\pi_\alpha^\gamma : \gamma X \to \alpha X$ is surjective. Thus, $\alpha X \leq \gamma X$. ∎

Proposition 13 *$K(X)$ is a complete lattice if and only if X is locally compact.*

Proof: We will prove the statement that if X is locally compact then $K(X)$ is a complete lattice, leaving the other direction to the reference [Ch].

Notice that $K(X)$ is always upper complete, so we only need to show that it is also lower complete. Thus let $\{\alpha_i\} \subset K(X)$. We want to produce $\alpha X \in K(X)$ so that $\alpha X \leq \alpha_i X$ for every i and it is a maximum with respect to this property. Consider $C = \bigcap C_{\alpha_i}$. Since each C_{α_i} is a closed unital algebra, so is C. Furthermore, $C_0(X) \subset C_{\alpha_i}$ for all i and since X is locally compact, $C_0(X)$ separates points from closed sets. Thus $C_0(X) \subset C$ so C separates points from closed sets. Let αX be the compactification you get from either construction by using C, then $C = C_\alpha$ and thus by Theorem 12 $\alpha X \leq \alpha_i X$ for all i. We claim that αX is the greatest lower bound of $\{\alpha_i X\}$. To show this, suppose that $\gamma X \leq \alpha_i X$ for all i. Then $C_\gamma \subset C_{\alpha_i}$ for all i so $C_\gamma \subset C = C_\alpha$ thus $\gamma X \leq \alpha X$ by Theorem 12. ∎

Another nice property of locally compact spaces is that $\alpha X \setminus X$ is closed for every $\alpha X \in K(X)$. This is an important property that makes the locally compact case very nice.

Proposition 14 *If X is locally compact then $\alpha X \setminus X$ is closed for any $\alpha X \in K(X)$.*

Proof: Since X is locally compact, it has a one-point compactification – ωX. By Proposition 11 there is some $\pi_\omega^\alpha : \alpha X \to \omega X$ with $\pi_\omega^\alpha|_X = id_X$. This means that $\pi_\omega^\alpha(\alpha X \setminus X) = \omega X \setminus X$ or $(\pi_\omega^\alpha)^{-1}(\omega X \setminus X) = \alpha X \setminus X$. Since $\omega X \setminus X$ is a point, it is closed. This means that $\alpha X \setminus X$ is also closed, being the inverse image of a closed set under a continuous map. ∎

In most cases it is impossible to determine $K(X)$. However, in some special cases it is is possible. We give an example of a family of such cases.

Example

Let α be an ordinal number. Define

$$W(\alpha) = \{\sigma | \sigma < \alpha,\ \sigma \text{ is an ordinal } \}.$$

We put the order topology on $W(\alpha)$ (a base for the topology is all sets of the form $\{\sigma | \gamma < \sigma < \theta\} \subset W(\alpha)$), making $W(\alpha)$ into a Tychonoff space (see [Ch] for the details).

Proposition 15 *$W(\alpha)$ is compact if and only if α is not a limit ordinal.*

Proof: See [Ch] page 34.

Let ω_1 be the smallest uncountable ordinal and, more generally, let ω_α be the smallest ordinal of cardinality $\aleph_\alpha$.

Theorem 16 *Each $f \in C^*(W(\omega_\alpha))$ has a continuous extension to $W(\omega_\alpha + 1)$ if $\alpha \geq 1$.*

Proof: See [Ch] page 35.

Corollary 17 *$\beta W(\omega_\alpha) \sim W(\omega_\alpha + 1)$.*

Proof: Clearly $W(\omega_\alpha)$ embeds in $W(\omega_\alpha+1)$ and by Proposition 15, $W(\omega_\alpha+1)$ is compact. Since $W(\omega_\alpha+1) \setminus W(\omega_\alpha) = \{\omega_\alpha+1\}$ is a limit point of $W(\omega_\alpha)$ in $W(\omega_\alpha+1)$ we know $W(\omega_\alpha)$ is dense in $W(\omega_\alpha+1)$, so $W(\omega_\alpha+1)$ is a compactification of $W(\omega_\alpha)$. By Theorem 16, every continuous bounded real-valued function on $W(\omega_\alpha)$ extends to $W(\omega_\alpha+1)$. Thus by Theorem 7, $W(\omega_\alpha+1) \sim \beta W(\omega_\alpha)$. ∎

The above corollary tells us that the only compactification of $W(\omega_\alpha)$ is the one-point compactification (which is the Stone-Čech compactification in this case) if $\alpha \geq 1$. Thus $K(W(\omega_\alpha))$ is trivial, with just one element.

2.5 K(X) and Algebras of Functions

In this section we point out the nice relationship between $K(X)$ and the collection of all closed unital subalgebras of $C^*(X)$ which separate points from closed sets. This section is really just collecting results which we have already proved.

Suppose we have $\mathcal{A} \subset C^*(X)$, a closed unital subalgebra which separates points from closed sets. Then using either construction, we get a compactification $\alpha X = e_{\mathcal{A}} X$. Furthermore, we know that $C_\alpha = \mathcal{A}$ from Proposition 3 (or the set of functions in $C^*(X)$ which extend to αX is $\mathcal{A}$).

Conversely, if we have a compactification αX of X, consider the algebra $\mathcal{A} = \{f|_X | f \in C^*(\alpha X)\}$. This algebra is closed and unital. Suppose that $x \in X$ and we have a closed $C \subset X$ with $x \notin C$, then $x \notin cl_{\alpha X}(C)$ so there is some $f \in C^*(\alpha X)$ with $f(x) = 0$ and $f(cl_{\alpha X}(C)) = 1$. Therefore, $f|_X(x) = 0$ and $f|_X(C) = 1$ so $f|_X$ separates x from C and, thus $\mathcal{A}$ separates points of from closed sets in X.

The above relationship is actually an isomorphism of partially ordered sets, as we show now.

Theorem 18 *The partially ordered set $K(X)$ is order isomorphic to the partially ordered set of all closed unital subalgebras of $C^*(X)$ which separate points from closed sets.*

Proof: The isomorphism is the functions Ψ which takes $\alpha X \in K(X)$ to C_α. We will prove that it is an isomorphism now.

First, Ψ is order-preserving by Theorem 12. Suppose that $C_\alpha = C_\gamma$ for two compactification αX and γX. Then, again by Theorem 12, we know that $\alpha X \sim \gamma X$. Thus Ψ is injective. Finally, suppose that $\mathcal{A}$ is a closed unital subalgebra of $C^*(X)$ which separates points from closed sets. Then we construct the compactification $e_{\mathcal{A}}X$ from $\mathcal{A}$ and we know from Proposition 1 that the set of all functions in $C^*(X)$ which extend to $e_{\mathcal{A}}X$ is $\mathcal{A}$. Thus the image of $e_{\mathcal{A}}$ under Ψ is $\mathcal{A}$, or Ψ is surjective. So Ψ is an isomorphism. ∎

This isomorphism allows one to translate between statements about compactifications and statements about closed unital algebras. Sometimes it is more illuminating to view a problem in the context of algebras and sometimes it is more illuminating to view a problem in the context of compactifications.

We define a very useful concept now. Let S and T be sets with a function $\phi : S \to T$. Then ϕ induces an algebra homomorphism $\phi^* : \mathbb{R}^T \to \mathbb{R}^S$ defined by $\phi^*(f)(s) = f(\phi(s))$ for all $f \in \mathbb{R}^T$. We call this the *pull-back* of ϕ. If ϕ is injective, then ϕ^* will be surjective. Conversely, if ϕ is surjective, then ϕ^* will be injective.

Let $\alpha X, \gamma X \in K(X)$ with $\alpha X \leq \gamma X$, so there is a quotient map $\pi_\alpha^\gamma : \gamma X \to \alpha X$. Then we know that $C_\alpha \subset C_\gamma$. Both C_α and C_γ are closed unital algebras so there is a inclusion of C_α into C_γ, let us call this inclusion i. Then we can prove that $i = (\pi_\alpha^\gamma)^*$ (restricted to the appropriate space), so there is a very natural relationship between the function you get in the order on $K(X)$ and in the order on the closed unital algebras. Another way of saying this is that C_α is the set of all functions $f \in C_\gamma$ so that $f^\gamma|_{(\pi_\alpha^\gamma)^{-1}(y)}$ is constant for all $y \in \alpha X$. We prove this statement now.

Proposition 19 *Suppose $C_\alpha \subset C_\gamma$ and let $i : C_\alpha \to C_\gamma$ be the inclusion. Then $i = (\pi_\alpha^\gamma)^*$.*

Proof: We know that C_α is isomorphic to $C^*(\alpha X)$ and similarly $C_\gamma \cong C^*(\gamma X)$ (see the diagram below). Since π_α^γ is a surjection, $(\pi_\alpha^\gamma)^*$ is an injection. Further, for each $f \in C^*(\alpha X)$ and $x \in X$ we have $(\pi_\gamma^\alpha)^*(f)(x) = f(\pi_\gamma^\alpha(x)) = f(x)$ so $(\pi_\gamma^\alpha)^*(f)|_X = f|_X$, which means that $(\pi_\gamma^\alpha)^*$ preserves the function values on X. This and the natural isomorphisms from Proposition 1 prove the result. ∎

$$\begin{tikzcd}
C^*(\alpha X) \arrow[r, "(\pi_\alpha^\gamma)^*"] \arrow[d, "\cong"'] & C^*(\gamma X) \arrow[d, "\cong"] \\
C_\alpha \arrow[r, "i"] & C_\gamma
\end{tikzcd}$$

Figure 4: Figure for Proposition 19

2.6 Remainder Considerations

Let $\alpha X \in K(X)$. Then the set $\alpha X \setminus X$ is called a *remainder* of X. The collection

$$Rem(X) = \{\alpha X \setminus X | \alpha X \in K(X)\}$$

is the collection of all remainders of X. There are some nice properties of $Rem(X)$.

For one thing, if X is locally compact then every element of $Rem(X)$ is closed (thus compact). This is just the statement of Proposition 14.

Another nice property is that if X is locally compact, then any image of a remainder is a remainder.

Theorem 20 *Let X be locally compact and $\alpha X \in K(X)$. Suppose that Y is such that there is some $\phi : \alpha X \setminus X \to Y$ with ϕ a surjection. Then $Y \in Rem(X)$.*

Proof: See [MG1] Theorem 2.1.

The following diagram illustrates the basic idea. We use $\coprod$ to signify the topological sum. Notice that not only is $Y \in Rem(X)$ but also we have that $\gamma X = Y \coprod X \leq \alpha X$.

$$\begin{tikzcd}
\alpha X \setminus X \arrow[d, "\phi"'] & \coprod & X \arrow[d, "id_X"] \\
Y & \coprod & X
\end{tikzcd}$$

Figure 5: Figure for Theorem 20

The proof in [MG1] first constructs $\gamma X = X \coprod Y$ (the disjoint union) and then topologizes γX with the quotient topology by the map $\pi_\gamma^\alpha : \alpha X \to \gamma X$ (the map is as in the diagram above). Another way to describe the topology on γX is using real-valued functions. Define $\mathcal{A} = \{f \in C^*(\alpha X) |\ f|_{\phi^{-1}(y)}$ is constant $\forall y \in Y\}$, then we can think of $\mathcal{A}$ as an algebra of functions on γX. It turns out that the weak topology by $\mathcal{A}$ is the same as the quotient topology by π_γ^α (this is due in part to the fact that $\mathcal{A}$ separates the points of γX and in part to Proposition 19).

Chapter 3

Countable Compactifications and a Generalization of the Hahn-Mazurkeiwicz Theorem

Theorems which guarantee the existence of a mapping of certain type are generally useful. A prime example is Urysohn's Lemma, which is used constantly in topology – for example in the study of $C(X)$ when you want to construct a continuous real-valued function with certain properties. Another example of this type theorem is the theorem which states that any compact metric space is the continuous image of the Cantor space. For instance, one of the uses of this theorem is in Banach space theory where you use the theorem to show that any separable Banach space can be viewed as a subspace of $C[0,1]$.

The Hahn-Mazurkeiwicz Theorem is another example of this. For reference, here is the statement of the theorem (from [W], Theorem 31.5):

Theorem 21 *(Hahn-Mazurkeiwicz) A Hausdorff space X is a continuous image of $[0,1]$ if and only if X is a compact, connected, locally connected metric space.*

This theorem is remarkable since (like the theorem about compact metric spaces and the Cantor space) it guarantees a map from a single space to a whole class of spaces. However, it is trivial to show that there is a surjective mapping from a (non-singleton) connected compact metric space onto $[0,1]$. So we really get a result about the existence of a surjection between any two compact, connected, locally connected metric spaces (just compose the two maps).

One kind of generalization of the Hahn-Mazurkeiwicz Theorem is Theorem 5.4 in [CChF]. This theorem guarantees the existence of a surjection between any two locally connected

generalized continua with the n-complementation property (definition 8). One way to interpret this theorem is that for a locally connected metric continuum (also called a *Peano* space) we can specify the images of a finite set of points, provided that when we take the points away from the space what remains is still connected. So this is one way that it is a generalization of the Hahn-Mazurkeiwicz Theorem. Another way to view this theorem is that you extend the Hahn-Mazurkeiwicz Theorem to non-compact spaces but you need to have the same "number of infinities" to match them up in order to get a perfect map.

The main result in this chapter is Theorem 36 which extends Theorem 5.4 in [CChF] to countably many points. When you have countably many points, you need to worry about how these points cluster. Unlike the finite case, there can be non-trivial topology on countably many points. This is where the classification of countable compact Hausdorff spaces (by Mazurkeiwicz and Sierpinski in [MS]) comes in. Roughly, we prove that as long as the types (of these countably many points) match, you can construct a mapping from one space to the other – see Theorem 36 for a precise statement of this.

A *generalized continuum* is a locally compact, connected, separable metric space.

3.1 Countable Compactifications

Let H be a countable compact Hausdorff space, then there is a countable ordinal α and an integer $n > 0$ so that H is homeomorphic to the disjoint union of n copies of ω^α (where we give ω^α the order topology) [MS]. In this case, we say that H is of type (α, n). What we get is that $H^{(\alpha)} = \{x_1, x_2, \ldots, x_n\}$ where $H^{(\alpha)}$ is the derived set of H of order α.

Let γX be a compactification of X. We say that γX is a *countable compactification* if $\gamma X \setminus X$ is countable. We say that γX is a *countable compactification of type* (α, n) if $\gamma X \setminus X$ is of type (α, n).

If we have two positive integers n and m and two countable ordinals α and γ, then we say that $(\alpha, n) \leq (\gamma, m)$ if either $\alpha < \gamma$ or $\alpha = \gamma$ and $n \leq m$.

For a countable ordinal α and a positive integer n, we define the (α, n) complementation property (definition 8). This property is a generalization of the n complementation property

in [C1].

In order to make this definition, we first need to recall the definition of the complementation degree of a locally compact Hausdorff space from [C2]. The complementation degree of X is denoted by $\Psi(X)$.

Definition 7 *We define $\Psi(X) \geq \alpha$ inductively. If X is compact, then $\Psi(X) = -1$. If X is not compact, $\Psi(X) \geq 0$. Let α be an ordinal and suppose we have defined $\Psi(X) \geq \sigma$ for every $\sigma < \alpha$. Then we say that $\Psi(X) \geq \alpha$ if for every $\sigma < \alpha$ and for every integer n there exist pairwise disjoint open sets $G_1, G_2, \ldots, G_n$ such that each has a compact boundary and $\Psi(cl(G_i)) \geq \sigma$ for all $i = 1 \ldots n$.*

If $\Psi(X) \geq \alpha$ and it is not true that $\Psi(X) \geq \alpha + 1$ then we say that $\Psi(X) = \alpha$.

Notice that it is possible for $\Psi(X) \geq \alpha$ for every α. For example, $\Psi(I\!N) \geq \alpha$ for every α. However, $I\!N$ is not a locally connected generalized continuum, so this example is not very interesting for us. The following is an example of a locally connected generalized continuum X with $\Psi(X) \geq \alpha$ for every α.

Example Let $E_0 = \{(0,0)\} \subset I\!R^2$. Let $E_{n+1} = \{(x \pm 2^{-n+1}, y + 2^{-n+1}) \mid (x,y) \in E_n\}$. Now let X be $\bigcup_n E_n$ and all the line segments joining a point in E_n with it's two successors in E_{n+1} (the successors of the point $(x,y) \in E_n$ are the two points $(x + 2^{-n+1}, y + 2^{-n+1})$ and $(x - 2^{-n+1}, y + 2^{-n+1})$). Then $\Psi(X) \geq \alpha$ for every α. This is because for any $n \in I\!N$, there are n disjoint subspaces $X_i \subset X$ so that each X_i is homeomorphic to X. This is the crucial property which makes $\Psi(X) \geq \alpha$ for all α. $cl(X) \subset I\!R^2$ is a dendrite (see next section for a definition and discussion of dendrites) with endpoints homeomorphic to the Cantor set.

The following three results from [C2] give a good indication of the meaning of $\Psi(X) \geq \alpha$. We will also need these results.

Proposition 22 *(Theorem 8,[C2]) A locally compact Hausdorff space X has a countable compactification of type (α, n) for some n if and only if $\Psi(X) \geq \alpha$.*

Corollary 23 *(Theorem 9, [C2]) Suppose that X has a maximal countable compactification of type (α, n). Then $\Psi(X) = \alpha$.*

Proposition 24 *(Theorem 10, [C2]) Suppose that $\Psi(X) = \alpha$. Then X has a maximal countable compactification of type (α, n) for some n or X has a non-metric uncountable totally disconnected compactification.*

A map $f : X \to Y$ is *perfect* if it is a closed continuous map so that $f^{-1}(y)$ is compact for every $y \in Y$.

This implies that $f^{-1}(K)$ is compact for every compact $K \subset Y$ ([PW], 1.8(d)).

We comment that if $f : X \to Y$ with X compact and Y Hausdorff, then f is perfect. This simple fact is very useful for us.

Proposition 25 *Let $f : X \to Y$ be a perfect surjection. If $\Psi(Y) \geq \alpha$ then $\Psi(X) \geq \alpha$.*

Proof: Induction

Basis ($\alpha = 0$) If Y is not compact, then X is not compact since f is a surjection.

Induction step. Suppose that the proposition is true for all $\sigma < \alpha$. Let $\sigma < \alpha$ and $n \in I\!N$ be given. Choose $H_1, H_2, \ldots, H_n \subset Y$ disjoint and open with $bd(H_i)$ compact and $\Psi(cl(H_i)) \geq \sigma$. Then $G_i = f^{-1}(H_i)$ are open, disjoint with compact boundaries. By the induction hypothesis $\Psi(cl(G_i)) \geq \sigma$ ($f|_{cl(G_i)}$ is perfect and $f^{-1}(cl(H_i)) = cl(G_i)$). ∎

Lemma 26 *Let $A \subset X$ be a closed subset with $bd(A)$ compact and $cl(int(A)) = A$. If $\Psi(A) \geq \alpha$ then $\Psi(X) \geq \alpha$.*

Proof: Induction

Basis ($\alpha = 0$) Clearly if $\Psi(A) \geq 0$ then $\Psi(X) \geq 0$ since a closed subset of a compact space is compact.

Induction step. Suppose it is true for all $\sigma < \alpha$.

Choose $\gamma < \alpha$ and $n \in I\!N$.

If α is a limit ordinal, then there is some σ with $\gamma < \sigma < \alpha$. By the induction hypothesis, we are done.

Thus, suppose that $\alpha = \sigma + 1$. Choose $H_1, H_2, \ldots, H_n \subset A$ open in A, disjoint with $bd_A(H_i)$ compact and $\Psi(cl(H_i)) \geq \sigma$. Let $G_i = H_i \cap int_X(A)$. Then $cl(G_i) = cl(H_i)$, since $cl(int(A)) = A$. ∎

Lemma 27 *Let $A = cl(X \setminus K)$ where K is compact. Then $\Psi(X) \geq \alpha$ if and only if $\Psi(A) \geq \alpha$.*

Proof: The proof is by induction on α.

Basis ($\alpha = 0$) If A is not compact, then clearly X is not compact. Conversely, if A is compact then clearly X is also compact. Thus, $\Psi(X) \geq 0$ if and only if $\Psi(X) \geq 0$.

Induction step. Now, suppose that both A and X are not compact

Clearly $int(A) \neq \emptyset$ (since X is not compact). Since $A = cl(X \setminus K)$, $cl(int(A)) = A$. Thus by Lemma 26 if $\Psi(A) \geq \alpha$ then $\Psi(X) \geq \alpha$.

So, suppose that $\Psi(X) \geq \alpha$. Let $\sigma < \alpha$ and $n \in I\!N$. Then there are $H_1, H_2, \ldots, H_n \subset X$ open and disjoint with compact boundaries and $\Psi(cl(H_i)) \geq \sigma$. Let $G_i = A \cap H_i$. Then $cl(G_i) = cl(cl(H_i) \setminus K)$ so by the induction hypothesis, $\Psi(c(G_i)) \geq \sigma$. Clearly the G_i's are open, disjoint subsets of A with compact boundaries. ∎

Using the preceeding two lemmas, we can prove the following stronger version of Lemma 26.

Proposition 28 *Let $A \subset X$ be closed with compact boundary. If $\Psi(A) \geq \alpha$ then $\Psi(X) \geq \alpha$.*

Proof: Clearly the proposition is true for $\alpha = 0$.

Let $B = cl(int(A))$. Then $int(B) \subset int(A)$ so $B = cl(int(A)) \subset cl(int(B))$, so $B = cl(int(B))$. Now notice that $B = cl(A \setminus bd(A))$. Thus, if $\Psi(A) \geq \alpha$, then $\Psi(B) \geq \alpha$ by Lemma 27. However, then Lemma 26 implies that $\Psi(X) \geq \alpha$. ∎

And now here is another useful property.

Proposition 29 *If $\Psi(X) < \alpha$ and α is a limit ordinal, then there is some $\sigma < \alpha$ so that $\Psi(X) < \sigma$.*

Proof: Suppose not. Then $\Psi(X) \geq \sigma$ for all $\sigma < \alpha$. We apply the definition of $\Psi(X) \geq \alpha$. Let $\sigma < \alpha$ and $n \in I\!N$. Then $\sigma + 1 < \alpha$ so there exists pairwise disjoint open sets $G_1, G_2, \ldots, G_n$ such that each G_i has compact boundary and $\Psi(cl(G_i)) \geq \sigma$. Thus $\Psi(X) \geq \alpha$, a contradiction. ∎

If you interpret Ψ as measuring how "complicated" X is, the preceeding results are not surprising. For instance, Proposition 28 states that if X has a closed subset which is "complicated", then X must be "complicated".

We now give the definition of the (α, n) complementation property.

Definition 8 *Let α be a countable ordinal and n be a positive integer. We say that X has the* (α, n) complementation property *if given any closed set $A \subset X$ with* bd*(A) compact and $\Psi(A) < \alpha$, there is a closed set $F \supset A$ with* bd*(F) compact and $\Psi(F) < \alpha$ such that $X \setminus F = \cup_1^n G_i$, where the G_i's are pairwise disjoint open connected sets with each $\Psi(cl(G_i)) \geq \alpha$.*

Roughly speaking, the (α, n) complementation property measures the "number" and "clustering" of the points at infinity in X. Theorem 35 makes this precise. Notice that the $(0, n)$ complementation property is the n complementation property from [CChF].

There are useful properties of the (α, n) complementation property which correspond to those of Ψ.

Proposition 30 *Let $A \subset X$ is closed subset with $bd(A)$ compact and $cl(int(A)) = A$. If A has the (α, n) complementation property and X has the (γ, m) complementation property then $(\alpha, n) \leq (\gamma, m)$.*

Proof: This is a relatively straightforward exercise in using the definitions and Lemma 26. ∎

Proposition 31 *Let $f : X \to Y$ be a perfect surjection. If Y has the (α, n) complementation property and X has the (γ, m) complementation property then $(\alpha, n) \leq (\gamma, m)$.*

Proof: This is also a relatively straightforward exercise in using the definitions and Proposition 25. ∎

Proposition 32 *Let $A = cl(X \backslash K)$ for compact K. Then A has the (α, n) complementation property if and only if X has the (α, n) complementation property.*

Proof: This is also a rather straightforward exercise in using the definitions and Lemma 27. ∎

Proposition 33 *Let X have the (α, n) complementation property and $A \subset X$ be a closed subset of X with $bd(A)$ compact. Suppose A has the (γ, m) complementation property. Then $(\gamma, m) \leq (\alpha, n)$.*

Proof: This just combines Propositions 30 and 32 similar to the way Proposition 28 combined Lemmas 26 and 27. ∎

Now the characterization of when a locally connected generalized continuum has a countable compactification of type (α, n). This is similar to Theorem 3.4 in [C1].

We need Proposition 3.12 of [MC1] for the proof of the next theorem, so we state it for reference.

Lemma 34 *(Proposition 3.12, [MC1]) If we remove a zero-dimensional subset S from a compact Hausdorff locally connected space Y, leaving a dense subspace $X = Y \setminus S$, then the*

restoration of S to Y gives us Y as the maximum zero-dimensional compactification of X, if and only if every connected open neighborhood in Y of any point of S remains connected when we remove S.

Theorem 35 *Let X be a locally connected generalized continuum. Then X has a maximal countable compactification of type (α, n) if and only if X has the (α, n) complementation property.*

Proof: Suppose that X has the (α, n) complementation property. We wish to show that X has a maximal countable compactification of type (α, n).

First we show the existence of a compactification of type (α, n). Let F and G_i be as in the definition of the (α, n) complementation property (with $A = \{x\}$ for some $x \in X$). Since $\Psi(cl(G_i)) \geq \alpha$, by Proposition 22, there is a compactification $\alpha(cl(G_i))$ of type $(\alpha, 1)$. Let $\alpha(F)$ be the one-point compactification of F. Then there is a countable compactification αX so that $\alpha X \setminus X = \{p\} \cup \bigcup_{i=1}^{n} \alpha(cl(G_i)) \setminus cl(G_i)$ and this αX is of type (α, n).

Now we show that there can be no compactification γX of type $(\gamma, m) > (\alpha, n)$. Suppose that there were such a compactification. Without loss of generality, we suppose that $\gamma = \alpha$ and $m = n + 1$. Let $\{x_1, x_2, \ldots, x_{n+1}\} = (\gamma X \setminus X)^{(\alpha)}$. Choose open neighborhoods H_i of x_i so that $bd(H_i)$ is compact and $H_i \cap H_j = \emptyset$ for $i \neq j$. Let $A = X \setminus \bigcup_{i=1}^{n+1} H_i$. Then A is a closed set with compact boundary and $\Psi(A) < \alpha$ (we know that $\Psi(A) < \alpha$ since $x_i \notin A$ for each i). We claim that there is no closed set $F \supset A$ with compact boundary and $\Psi(F) < \alpha$ so that $X \setminus F = \bigcup_{i=1}^{n} G_i$ where G_i's are pairwise disjoint open connected set with compact boundary and $\Psi(G_i) \geq \alpha$.

Suppose there were such a set $F \supset A$. Then $X \setminus F \subset X \setminus A$ so $\bigcup_{i=1}^{n} G_i \subset \bigcup_{i=1}^{n+1} H_i$. Since the G_i's are connected, there is some H_i so that $H_i \cap G_j = \emptyset$ for all j. Let this be H_1.

However, this implies that $cl(H_1) \subset F$. Since H_1 has a compactification of type $(\alpha, 1)$, $\Psi(H_1) \geq \alpha$. But then Proposition 28 implies that $\Psi(F) \geq \alpha$ which is a contradiction. Thus, no such F can exist. However, this contradicts the fact that X has the (α, n) complementation property. Thus, no such compactification γX can exist.

Conversely, suppose that X has a maximal countable compactification αX of type (α, n).

Let $\{x_1, x_2, \ldots, x_n\} = (\alpha X \setminus X)^{(\alpha)}$. Suppose that $A \subset X$ is closed with compact boundary and $\Psi(X) < \alpha$. By Theorem 2.8 and Proposition 3.12 of [MC1] (34), $cl(A)$ is the maximum countable compactification of A and by Proposition 22 $(cl(A) \setminus A)^{(\alpha)} = \emptyset$. Furthermore, since A has compact boundary $cl(A) \setminus A$ is a closed subset of $\alpha X \setminus X$ and thus $(cl(A) \setminus A)^{(\alpha)} = cl(A) \cap (\alpha X \setminus X)^{(\alpha)}$. Thus, $x_i \notin cl(A) \setminus A$ for each i. Choose H_i an open connected neighborhood of x_i with $bd(H_i) \cap \alpha X \setminus X = \emptyset$ and $H_i \cap H_j = \emptyset$ for $i \neq j$. Furthermore, we require that $H_i \cap cl(A) = \emptyset$ for each i. Let $G_i = X \cap H_i$ then by Proposition 3.12 in [MC1], G_i is connected. Let $F = X \setminus \bigcup_{i=1}^{n} G_i$. Then $F \supset A$ is a closed set with compact boundary and $\Psi(F) < \alpha$, since $x_i \notin cl(F)$ for each i. We know that $(H_i \setminus G_i)^{(\alpha)} = \{x_i\}$ so $H_i \cup bd(H_i)$ is a countable compactification of G_i of type $(\alpha, 1)$ and thus $\Psi(cl(G_i)) \geq \alpha$ (here we are thinking of the closure of G_i as a subset of X). Therefore, X has the (α, n) complementation property. ∎

So, what the (α, n) complementation property measures is the "clustering" of the "components of ∞". Another way to see this is that in the maximal totally disconnected compactification, each component of $\beta X \setminus X$ has been quotiented to a point. Thus, the (α, n) complementation property measures the way the components of $\beta X \setminus X$ cluster. There is no nice way to measure the "clustering" of the components of $\beta X \setminus X$ if there are uncountably many components. Thus, this result is restricted to countably many components.

Notice that in the special case of locally connected generalized continua, we can use Theorem 35 to prove Propositions 30, 31, and 32 (none of which are used in the proof of Theorem 35).

3.2 Mappings of Locally Connected Generalized Continua

The Hahn-Mazurkeiwicz Theorem characterizes those spaces which are images of the unit interval. The main result of this section can be thought of as one kind of generalization of the Hahn-Mazurkeiwicz Theorem, where we relax the requirements on the space or, alternatively, where we specify the image of a countable compact subspace.

Throughout the rest of this chapter X and Y are locally connected generalized continua with the (α, n) complementation property. Thus both X and Y have a maximal countable compactification of type (α, n). Let αX and αY denote these compactifications. Since $\alpha X \setminus X$ is countably infinite, it is not perfect. Thus by 5.2(c) of [NA], αX is locally connected. Similarly, αY is connected and locally connected.

Here is the main result. The next several sections will present the proof.

Theorem 36 *Let n be a positive integer and α be a countable ordinal. Suppose that X and Y are two locally connected generalized continua with the (α, n) complementation property. Then there is a perfect surjection $f : X \to Y$.*

We can also state the main result without reference to the (α, n) complementation property.

Theorem 37 *Let α be a countable ordinal and $n > 0$ be an integer. Then if X and Y are any two spaces whose maximal countable compactifications are of type (α, n), then there is a perfect surjection $f : X \to Y$.*

Notice that this theorem is equivalent to the previous one by Theorem 35.

Using this theorem, we get two perfect surjections $f : X \to Y$ and $g : Y \to X$. However, we prove something a bit stronger. We will prove that the maps f and g extend to maps $f^\alpha : \alpha X \to \alpha Y$ and $g^\alpha : \alpha Y \to \alpha X$ with $(f^\alpha)^{-1}(\alpha Y \setminus Y) = \alpha X \setminus X$ and $(g^\alpha)^{-1}(\alpha X \setminus X) = \alpha Y \setminus Y$. Here αX and αY are the maximal countable compactifications of X and Y respectively.

This leads us to think of Theorem 36 in the following way. Suppose that X and Y are Peano continua and $K \subset X$ and $C \subset Y$ are countable and compact with $X \setminus K$ and $Y \setminus C$ connected and locally connected. Then we can find surjections $f : X \to Y$ and $g : Y \to X$ so that $f^{-1}(C) = K$ and $g^{-1}(K) = C$. The Hahn-Mazurkeiwicz Theorem only gives us a surjection from X onto Y. Theorem 36 allows us to specify the image of a countable compact set as long as its complement is connected and locally connected. If $X \setminus K$ is disconnected, then clearly there is a space Y with $Y \setminus C$ connected so that $Y \setminus C$ cannot be

mapped onto $X \setminus K$. Thus $X \setminus K$ being connected is necessary. Furthermore, since every quotient of a locally connected space is locally connected (and $f|_{X\setminus K}$ is a quotient map), we know that local connectivity of $X \setminus K$ is also necessary. Theorem 36 states that the conditions that $X \setminus K$ be connected and locally connected are also sufficient.

We prove Theorem 36 by constructing spaces $T(X)$ and $T(Y)$ and factoring the desired mapping through these spaces ($X \to T(X) \to T(Y) \to Y$). The spaces $T(X)$ and $T(Y)$ have a particularly nice structure, which makes the constructions easier. In particular, we construct $T(X) \subset \mathbb{R}^2$ with $cl(T(X)) \subset \mathbb{R}^2$ a dendrite.

A *dendrite* is a Peano continuum which contains no simple closed curve. See chapter 10 of [NA] for more information on dendrites. We now give some of the most basic and useful (for us) properties of dendrites.

A dendrite is basically a generalization of a tree. In fact, Theorem 10.27 of [NA] tells us that any dendrite can be approximated (in an appropriate sense) by an increasing sequence of trees.

Every dendrite can be embedded in $\mathbb{R}^2$ (10.37 , [NA]). However, this will not be so important for us since we will explicitly construct our dendrites as subsets of $\mathbb{R}^2$.

Here is a theorem from [NA] which provides a characterization of dendrites among continua. We use this to prove that some of the spaces we construct are dendrites.

Theorem 38 *([NA], Theorems 10.2,10.7,10.10) A (nondegenerate) metric continuum X is a dendrite if and only if any of the following conditions hold:*

1. *Every two points in X are separated by a third point of X.*
2. *Each point of X is either a cut point or an end point.*
3. *The intersection of any two connected subsets is connected.*

We prove Theorem 36 in several stages. First, we construct the space $T(X)$ (which we prove to have the (α, n) complementation property) and show how we can map X onto $T(X)$ perfectly. Next we construct a perfect surjection $T(X) \to T(Y)$. The final step is to

show that we can map $T(Y)$ perfectly onto Y. We construct the spaces $T(X)$ and $T(Y)$ in a very natural way, which makes the maps $X \to T(X)$ and $T(Y) \to Y$ easy to construct. The hard work is in constructing the map $T(X) \to T(Y)$.

3.3 Construction of T(X)

Let $\{\mathcal{O}_n\}_{n=0}^{\infty}$ be any collection of open subsets of αX (the countable compactification of X of type (α, n)) with the following properties:

1. $\mathcal{O}_0 = X$
2. $cl(\mathcal{O}_{n+1}) \subset \mathcal{O}_n$
3. $\mathcal{O}_n = \bigcup_{i=1}^{m_n} O_{n,i}$ where $O_{n,i} \cap O_{n,j} = \emptyset$ if $i \neq j$ and $O_{n,i}$ is connected and open (thus, $\{O_{n,i}\}$ are the components of $\mathcal{O}_n$).
4. $\bigcap_{n=1}^{\infty} \mathcal{O}_n = \alpha X \setminus X$

For our purposes, we do not care what collection we choose, so long as it satisfies the above properties. For each different collection $\{\mathcal{O}_n\}$, we will get a different space $T(X)$. However, as we will see, this will not matter.

It is easy to see that such a collection exists for a given X. One way to construct $\{\mathcal{O}_n\}$ is to consider a countable neighborhood basis for each point of $\alpha X \setminus X$. Denote by $\{O(x, n)\}$ the neighborhood basis at x. Since $\alpha X \setminus X$ is compact, for each n there is a finite sub-collection of $\{O(x, n) | x \in \alpha X \setminus X\}$ which covers $\alpha X \setminus X$. Let $\mathcal{O}_n$ be the union of this finite sub-collection and let $\{O_{n,i}\}_{i=1}^{m_n}$ be the components of $\mathcal{O}_n$ (which are open since αX is locally connected). We know that there are only finitely many components since αX is compact. Furthermore, we can assume that $cl(\mathcal{O}_{n+1}) \subset \mathcal{O}_n$ since αX is normal. This collection satisfies the properties above.

We will need some further properties of the $\mathcal{O}_n$'s. The two propositions (and the corollary) will be used in the construction of the map $T(X) \to T(Y)$ and the two lemmas will be used to show that $\alpha X \setminus X$ is homeomorphic to $cl(T(X)) \setminus X$.

Notice that $cl(X \setminus \mathcal{O}_n)$ is compact in X for all n since $\alpha X \setminus X \subset \mathcal{O}_n$.

The following simple structure results are crucial to what follows.

Proposition 39 *Suppose that $\alpha X \setminus X$ is of type $(\alpha, 1)$. For any $\sigma < \alpha$ there is an $m \in I\!N$ so that*

$$\alpha X \setminus X = T \cup S_1 \cup \bigcup_{i=1}^{l} R_i$$

where each of T, S_1, R_j are in separate components of $\mathcal{O}_m$ and are closed in $\alpha X \setminus X$ with $T \cong \omega^\alpha$ and $S_1 \geq \omega^\sigma$ and $R_i < \omega^\alpha$.

Proof: Let $\{p\} = (\alpha X \setminus X)^{(\alpha)}$. Without loss of generality, we can assume that $p \in O_{k,1}$ for all $k \in I\!N$. If there is no such m, then $(\alpha X \setminus X) \setminus O_{k,1} < \omega^\sigma$ for all k so

$$(\alpha X \setminus X) \setminus \{p\} = \bigcup_k ((\alpha X \setminus X) \setminus O_{k,1}) \leq \omega^\sigma < \omega^\alpha.$$

However, this contradicts the fact that $\alpha X \setminus X$ is homeomorphic to $W(\omega^\alpha + 1)$. Thus, there is some $m \in I\!N$ so that

$$P_m \equiv \bigcup_{i \geq 2} ((\alpha X \setminus X) \cap O_{m,i}) = (\alpha X \setminus X) \setminus O_{m,1} \geq \omega^\sigma$$

Now clearly there must be some j so that $(\alpha X \setminus X) \cap O_{m,j} \geq \omega^\sigma$. Let $S_1 = (\alpha X \setminus X) \cap O_{m,j}$. Further, since $P_m < \omega^\alpha$, each $R_i = (\alpha X \setminus X) \cap O_{m,i} < \omega^\alpha$ for $i \neq j$. ∎

Corollary 40 *Suppose that $\alpha X \setminus X$ is of type $(\alpha, 1)$. For any $\sigma < \alpha$ and any $k \in I\!N$ there is an $m \in I\!N$ so that*

$$\mathcal{O}_m \cap \alpha X \setminus X \cong T \cup \bigcup_{i=1}^{k} S_i \cup \bigcup_{i=1}^{l} R_i$$

where each of T, S_i, R_j are in separate components of $\mathcal{O}_m$ and are closed in $\alpha X \setminus X$. Furthermore, $T \cong \omega^\alpha$ and $S_i \geq \omega^\sigma$ and $R_j < \omega^\alpha$.

Proof: One application of the proposition gives us one such set S_1. Now we just use the proposition again replacing $\alpha X \setminus X$ with T and we will get another such set S_2. We just continue this until we get the desired result. ∎

Proposition 41 *Suppose that $\alpha X \setminus X$ is of type (α, n). Then there is some $m \in I\!N$ so that*

$$\mathcal{O}_m \cap \alpha X \setminus X \cong \bigcup_{i=1}^{n} S_i \cup \bigcup_{i=1}^{l} R_i$$

where each of S_i and R_j are in separate components of $\mathcal{O}_m$ and are closed in $\alpha X \setminus X$. Furthermore, $S_i \cong \omega^\alpha$ for each $i = 1, \ldots, n$ and $R_j < \omega^\alpha$ for each $j = 1, \ldots, l$.

Proof: The proof of this proposition is similar to that of Proposition 39. The difference is that here $\alpha X \setminus X \cong W(n\omega^\alpha + 1)$ instead of $W(\omega^\alpha + 1)$. ∎

Lemma 42 *For each $x \in \alpha X \setminus X$, there is a unique sequence of sets $\{O_{n,x_n}\}_{n=1}^{\infty}$ such that $\{x\} = \bigcap_{n=1}^{\infty} O_{n,x_n}$.*

Proof: This is basically a consequence of the fact that $\alpha X \setminus X$ is totally disconnected and of property 4 above.

The existence of such a sequence is easy. For each n, let O_{n,x_n} be the component of $\mathcal{O}_n$ which contains x. Clearly there is only one such component since the $O_{n,i}$'s are pairwise disjoint.

Suppose that there is another such sequence $\{O_{n,\hat{x}_n}\}_{n=1}^{\infty}$. Then there is some n so that $O_{n,x_n} \neq O_{n,\hat{x}_n}$. This is a contradictions since $x \in O_{n,x_n}$ and $x \in O_{n,\hat{x}_n}$ while $O_{n,x_n} \bigcap O_{n,\hat{x}_n} = \emptyset$.∎

Lemma 43 *Let $\{O_n\}$ be such that O_n is a component of $\mathcal{O}_n$ and $cl(O_{n+1}) \subset O_n$. Then there is an $x \in \alpha X \setminus X$ so that $\{x\} = \bigcap_{n=1}^{\infty} O_n$.*

Proof: Let $x, y \in \bigcap_{n=1}^{\infty} O_n$. Let $K_n = cl(O_n)$ so that K_n is a metric continuum. The conditions on the O_n's insure that $K_{n+1} \subset K_n$, thus $\bigcap_{n=1}^{\infty} K_n$ is a metric continuum containing x and y. However, $\bigcap_{n=1}^{\infty} K_n \subset \bigcap_{n=1}^{\infty} O_n \subset \alpha X \setminus X$, which is totally disconnected. Thus, $x = y$.∎

These two lemmas imply that the points of $\alpha X \setminus X$ are in one-to-one correspondence with nested sequences from the $\mathcal{O}_n$'s.

We now proceed with the construction of $T(X)$ by induction. We will construct $T(X)$ as a subset of $I\!R^2$. First, we need to establish some notation.

Notation 9 *Let $v \in I\!R^2$ with $|v| = 1$, $v = (v_1, v_2)$.*

$$R(v) = \begin{pmatrix} v_2 & v_1 \\ -v_1 & v_2 \end{pmatrix}$$

For n an even positive integer:

$\mathcal{T}_n(l) = (\bigcup_{k=1}^{n/2} \{(x,y) | x = \pm \tan(k\pi/2n) y\}) \cap \{(x,y) | x^2 + y^2 < l^2, y \geq 0\}$

For n odd: $\mathcal{T}_n(l) = \mathcal{T}_{n-1}(l) \cup \{0\} \times [0, l]$

$\mathcal{T}_0 = (0,0)$

$\mathcal{T}_n(l, v) = R(v)\ \mathcal{T}_n(l)$

$\mathcal{T}_n(l)$ is just a spray of n lines with root at the origin and each of length l and symmetric about the y axis with a maximum spread of $\pi/2$ radians. $\mathcal{T}_n(l, v)$ is just $\mathcal{T}_n(l)$ rotated so that it is symmetric about the direction defined by v.

And now just a little bit more notation.

Notation 10 $I_n = \{0, 1, \ldots, m_n\}$

For n and i non-negative integers with $0 < i < m_n$:

$a(n,i) \in I_{n-1}$ *so that* $O_{n,i} \subset O_{n-1,a(n,i)}$.

$s(n,i) \subset I_{n+1}$ *so that* $j \in s(n,i)$ *implies* $O_{n+1,j} \subset O_{n,i}$.

$c(n) = \max\{|s(n,i)| \mid i \in I_n\}$

Recall that m_n is the number of components of $\mathcal{O}_n$.

We say that $a(n,i)$ is the ancestor function and $s(n,i)$ is the successor function.

We will also use the $\mathcal{T}_n(l, v)$ as the basic building-blocks in the construction of $T(X)$.

Inductive Construction of $T(X)$

$n = 1$:

$T_1(X) = \mathcal{T}_{c(0)}(1,(0,1))$

Step $n+1$:

$T_{n+1}(X) = T_n(X) \cup (\bigcup_{i \in I_n} T_{n+1,i}(X)$ where for all $i \in I_n$,

$$T_{n+1,i}(X) = v_{n,i} + \mathcal{T}_{|s(n,i)|}(l_{n+1}, \hat{v}_{n,i})$$

$v_{n,i}$ is the end-point of $T_n(X)$ corresponding to $O_{n,i}$.

$l_{n+1} = l_n \times \frac{1}{3} \times \frac{\pi}{8c(n)}$, with $l_0 = 1$

$\hat{v}_{n,i} = \frac{v_{n,i} - v_{n-1,a(n,i)}}{|v_{n,i} - v_{n-1,a(n,i)}|}$

End of Induction

Let

$$T(X) = \bigcup_{n=1}^{\infty} T_n(X).$$

The basic idea behind the construction is to make a "tree" which has the same branching structure as the collection $\{O_{n,i}\}$. The complications arise because we wish $T(X)$ to have compact closure in $\mathbb{R}^2$ and want $cl(T(X)) \setminus X$ to be homeomorphic to $\alpha X \setminus X$.

Notice that $T(X)$ depends both on X and on the collection $\{\mathcal{O}_n\}$. If you start with a different collection of open sets, you will get a different space $T(X)$.

We now investigate $T(X)$. This will give us some idea of what kind of space $T(X)$ is. We do not need all of these for the proof of the theorem, but they are helpful to illuminate what is going on.

Proposition 44 *$K \subset T(X)$ is closed if and only if $K \cap T_n(X)$ is closed for every $n \in \mathbb{N}$.*

Proof: One direction is trivial. Thus, suppose that $K \cap T_m(X)$ is closed for all $m \in \mathbb{N}$. Suppose that $x \in cl(K) \setminus K$. Then there is some $p \in \mathbb{N}$ so that $x \in int(T_p(X))$. This means that $K \cap T_p(X)$ is infinite, and $x \in cl(K) \cap T_m(M) = cl(K \cap T_m(X)) = K \cap T_m(X)$. Thus, $x \in K$. ∎

We wish to make rigorous the idea of "distance from the root". What we define is for two points $x, y \in T(X)$, $x \prec y$ if y separates x from the root. We also define $x \prec x$ to make

$\prec$ reflexive. Because of the way we constructed $T(X)$, it is easy to prove that $\prec$ is a partial order on the points of $T(X)$.

Notation 11 *Let* $R(X) = cl(T(X)) \setminus T(X)$.

If $v \in T(X)$, *then* $T(v, X) = \{x \in T(X) | v \prec x\}$

We warn the reader that $R(X)$ as defined here is not the same $R(X)$ as will be used in Chapter 4.

Notice that if $v \in T(X)$ is a vertex, then $T(v, X)$ is a space which has properties similar to those of $T(X)$ (This is clear from the inductive construction).

Lemma 45 *If* $C_1, C_2 \subset T(X)$ *are closed (in* $T(X)$*), connected, and disjoint, then* $cl_{\mathbb{R}^2}(C_1) \cap cl_{\mathbb{R}^2}(C_2) = \emptyset$.

Proof: First, we note that the lemma is trivially true if both C_1 and C_2 are both compact. So suppose that C_1 is compact and C_2 is not compact. Then there is an $n \in \mathbb{N}$ such that $C_1 \subset T_n(X)$ and $T_n(X) \cap C_2$ is compact. Clearly, $cl(C_2) \setminus C_2 \subset R(X)$ so $(cl(C_2) \setminus C_2) \cap C_1 = \emptyset$. Thus, we can suppose that neither C_1 nor C_2 are compact.

Claim: There is an $n \in \mathbb{N}$ and distinct vertices $v_{n,i_1}, v_{n,i_2}, \ldots, v_{n,i_m}, v_{n,j_1}, \ldots, v_{n,j_k}$ so that

$C_1 \setminus T_n(X) \subset T(v_{n,i_1}, X) \cup \cdots \cup T(v_{n,i_m}, X)$

$C_2 \setminus T_n(X) \subset T(v_{n,j_1}, X) \cup \cdots \cup T(v_{n,j_k}, X)$

Proof of Claim:

Let $v \in C_1$ be any vertex (which must exist since and C_1 is non-compact and closed). Choose a vertex $w \in C_2$. Then there is an $n \in \mathbb{N}$ so that $v \in T_{n-1}(X)$ and $w \in T_{n-1}(X)$. Since $T(X) \setminus T_n(X) = \bigcup\{T(v_{n,i}, X) | i \in \mathcal{I}_n\}$ and $C_1 \cap (T(X) \setminus T_n(X)) \neq \emptyset$ and $C_2 \cap (T(X) \setminus T_n(X)) \neq \emptyset$, there are vertices $v_{n,i_1}, v_{n,i_2}, \ldots, v_{n,i_m}, v_{n,j_1}, \ldots, v_{n,j_k}$ so that $C_1 \setminus T_n(X) \subset T(v_{n,i_1}, X) \cup \cdots \cup T(v_{n,i_m}, X)$ and $C_2 \setminus T_n(X) \subset T(v_{n,j_1}, X) \cup \cdots \cup T(v_{n,j_k}, X)$ and $C_1 \cap T(v_{n,i_a}, X) \neq \emptyset$ for $a = 1, \ldots, m$ and $C_2 \cap T(v_{n,j_a}, X) \neq \emptyset$ for $a = 1, \ldots, k$.

All we have left to prove is that $C_1 \cap T(v_{n,j_a}, X) = \emptyset$ for all $a = 1, \ldots, k$ and $C_2 \cap T(v_{n,i_a}, X) = \emptyset$ for all $a = 1, \ldots, m$. This will prove that the vertices are distinct.

Suppose that $C_1 \cap T(v_{n,j_a}, X) \neq \emptyset$. Since $w \in T_{n-1}(X)$, and $C_2 \cap T(v_{n,j_a}, X) \neq \emptyset$, there is a path $\sigma : [0,1] \to C_2$ so that $\sigma(0) = w$ and $\sigma(1) \in T(v_{n,j_a}, X)$. Thus, there is a $s \in (0,1)$ with $\sigma(s) = v_{n,j_a}$.

Similarly, there is a path $\epsilon : [0,1] \to C_1$ with $\epsilon(0) = v$ and $\epsilon(1) \in T(v_{n,j_a}, X)$ and and $s' \in (0,1)$ with $\epsilon(s') = v_{n,j_a}$. However, this contradicts the fact that $C_1 \cap C_2 = \emptyset$. Thus, the claim is proven.

By the previous two cases, we know that:

$(cl(C_1) \cap T_n(X)) \cap (cl(C_2) \cap T_n(X)) = \emptyset$

$(cl(c_1) \cap T_n(X)) \cap cl(C_2) = \emptyset$

and

$(cl(C_2) \cap T_n(X)) \cap cl(C_1) = \emptyset$.

We need only prove that $(cl(C_1) \setminus T_n(X)) \cap (cl(C_2) \setminus T_n(X)) = \emptyset$. However, this is clear since by the construction $cl(T(v_1, X)) \cap cl(T(v_2, X)) = \emptyset$ for any two distinct vertices v_1 and v_2 (this is due to the fact that separate branches are separated by a positive distance in $I\!R^2$). ∎

Lemma 46 *For each $x \in cl(T(X)) \setminus T(X)$, there is a unique arc $\sigma : [0,1] \to cl(T(X))$ so that $\sigma(0)$ is the root and $\sigma(1) = x$.*

Proof: Clearly $cl(T(X))$ is connected, compact, and metric so it is a continuum. Furthermore, $cl(T(X))$ is locally connected at each point of $T(X)$. Let

$$F = \{x \in cl(T(X)) | cl(T(X)) \text{ is not locally connected at } x\}.$$

Then $F \subset cl(T(X)) \setminus T(X)$. By 5.2 (c) of [NA], F is perfect and if $F \neq \emptyset$, contains a non-degenerate subcontinuum. However, $cl(T(X)) \setminus T(X)$ is totally disconnected. Thus $F = \emptyset$ so $cl(T(X))$ is locally connected. Therefore, $cl(T(X))$ is arcwise connected. Thus,

for each $x \in R(X)$, there is an arc $\sigma_x : [0,1] \to cl(T(X))$ from r_0 to x. Suppose that there are two such arcs (where we consider two arcs different if their ranges are different) σ and ϵ. Then there is an $s \in (0,1)$ and $s' \in (0,1)$ so that if $s < t < 1$ and $s' < t' < 1$ then $\sigma(t) \neq \epsilon(t')$. But then for some $\delta > 0$ $\sigma([s+\delta,1))$ and $\epsilon([s'+\delta,1))$ are two disjoint closed (in $T(X)$) sets so $cl(\sigma([s+\delta,1))) \cap cl(\epsilon([s'+\delta,1))) = \emptyset$ by Lemma 45. This contradicts the choice of σ and ϵ. ∎

Proposition 47 *$cl(T(X))$ is a dendrite.*

Proof: Clearly $cl(T(X))$ is compact and connected. Thus, we need only show that for every two distinct points $x, y \in cl(T(X))$, there is a point $z \in cl(T(X))$ which separates them in $cl(T(X))$.

If $x, y \in T(X)$, then there is a point $z \in T(X)$ which will separate them in $T(X)$, thus $T(X) \setminus \{z\} = C_1 \cup C_2 \cup \cdots \cup C_m$ where each C_i is connected. Without loss of generality, $x \in C_1$ and $y \in C_2$. Take $n \in \mathbb{N}$ so that $\{x, y, z\} \subset T_{n-1}(X)$, then $cl(C_1 \setminus T_n(X)) \cap cl(C_2 \setminus T_n(X)) = \emptyset$ (where these closures are taken in $T(X)$ and not in $\mathbb{R}^2$). All we need to show is that $cl(C_1 \setminus T_n(X)) \cap cl(C_2 \setminus T_n(X)) = \emptyset$. However, this is clearly so by Lemma 45.

So, now suppose that $x \in T(X)$ and $y \in R(X)$. Then by Lemma 46 there is a unique arc $\sigma : [0,1] \to cl(T(X))$ so that $\sigma(0) = r_0$ (the root) and $\sigma(1) = y$. Then there is an $s \in (0,1)$ so that $x \notin \sigma([s,1])$. Let $z = \sigma(s)$. Then z separates x from y. This is since z separates x from $\sigma([s,1))$ in $T(X)$ so the component C_1 of $T(X) \setminus \{x\}$ containing x is disjoint from the component C_2 containing $\sigma([s,1))$.

Finally, suppose that $x, y \in R(X)$ with $x \neq y$. Then there are different arcs $\sigma : [0,1] \to cl(T(X))$ and $\epsilon : [0,1] \to cl(T(X))$ with $\sigma(0) = \epsilon(0) = r_0$ and $\sigma(1) = x$ and $\epsilon(1) = y$. Let $s = \sup\{a \in [0,1) | \sigma(s) \in \epsilon([0,1])\}$, and $t = \sup\{a \in [0,1) | \epsilon(s) \in \sigma([0,1])\}$. Let $z = \sigma(s) = \epsilon(t)$, then z must be a branch point of $T(X)$ so, for some small $\delta > 0$, $\sigma([s+\delta,1))$ and $\epsilon([t+\delta,1)]$ are contained in two different components of $T(X) \setminus \{z\}$, say C_1 and C_2 respectively. Since they are different, $cl(C_1) \cap cl(C_2) \cap R(X) = \emptyset$ so z separates x and y. Therefore, any two points of $cl(T(X))$ can be separated by a third point. Hence,

$cl(T(X))$ is a dendrite. ∎

We also want to show that $T(X)$ has the (α, n) complementation property. However, we wait to do this until after we have constructed the map $X \to T(X)$.

3.4 Construction of a perfect map $f : X \to T(X)$

Now we will construct a perfect surjection $f : X \to T(X)$. We will use the notation from the preceeding section.

The construction of f will also be by induction. Let $y_0 \in X \setminus cl(\mathcal{O}_1)$.

Induction

$n = 1$:

Define $f_1 : X \setminus \mathcal{O}_1 \to T_1$ so that $f_1(y_0) = (0, 0)$ and $f_1^{-1}(v_{1,i}) = bd(O_{1,i})$. Such an f_1 exists since X is a normal space. Furthermore, f_1 so chosen will be surjective.

Step $n + 1$:

For each $i \in I_n$ define:

$f_{n+1,i} : cl(O_{n,i}) \setminus \mathcal{O}_{n+1} \to T_{n+1,i}$ so that:

$f_{n+1,i}(bd(O_{n,i})) = v_{n,i}$ and $f_{n+1,i}^{-1}(v_{n+1,i}) = bd(O_{n+1,j})$ for all $j \in s(n, i) \subset I_{n+1}$.

All of these maps can be defined using the normality of X. Furthermore, all of these maps will be surjective.

End of Induction

Now we define $f : X \to T(X)$ by:

$f|_{domain(f_{n,i})} = f_{n,i}$.

$f|_{X \setminus \mathcal{O}_1} = f_1$.

With this definition, f is continuous since each of the $f_{n,i}$'s are and they agree on the intersections of their domains (which are closed sets). Furthermore, f is surjective since each of the $f_{n,i}$'s are. Now we show that f is perfect.

Proposition 48 *f as defined above is a perfect surjection.*

Proof: Clearly f is surjective. Let $K \subset T(X)$ be compact. Then $K \subset T_n(X)$ for some large enough $n \in I\!N$. So $f^{-1}(K) \subset f^{-1}(T_n(X)) = X \setminus \mathcal{O}_n$ is compact in X.

Now let $C \subset X$ be closed. Then $f(C) \cap T_n(X) = f(K \cap (X \setminus \mathcal{O}_n))$ and $C \cap (X \setminus \mathcal{O}_n)$ is compact. Therefore, $f(C) \cap T_n(X)$ is compact hence closed. ∎

Thus we have our perfect surjection $f : X \to T(X)$.

We now wish to show that we can extend this map to a map $\alpha X \to cl(T(X))$. If we have a map $f^\alpha : \alpha X \to cl(T(X))$ with $f^{-1}(cl(T(X) \setminus T(X)) = \alpha X \setminus X$ then it is trivial that $f^\alpha|_X$ would be perfect, so this is an another way to show that the map $f : X \to T(X)$ is perfect. However, since our main interest is in the map $X \to T(X)$ and not in the map $\alpha X \to cl(T(X))$, we do not use this method to prove that f is perfect.

The only difficulty is defining the extension on points of $\alpha X \setminus X$. If we can do this in a natural way, it should be easy to show that the extension is continuous.

Let $x \in \alpha X \setminus X$. Then by Lemma 42, there is a unique sequence $\{O_{n,x_n}\}_{n=1}^\infty$ so that $x = \bigcap_n O_{n,x_n}$. Define $f^\alpha(x) = \lim_n f(x_n)$ where $x_n \in O_{n,x_n} \setminus cl(O_{n+1,x_{n+1}})$. Since $diam(T_{n+1}(X) \setminus T_n(X)) \to 0$, the limit exists and is independent of the choice of the points $\{x_n\}$. Clearly, this definition makes the extension continuous. Furthermore, by Lemmas 42 and 43, it is surjective.

Now we use this map to explore the structure of $T(X)$ further.

Let $U_{n,i} = f(O_{n,i})$. Clearly $U_{n,i}$ is open (since f is perfect) and $U_{n,i} \cap U_{n,j} = \emptyset$ if $i \neq j$. Furthermore, $U_{n,i} \subset U_{m,j}$ if and only if $O_{n,i} \subset O_{m,j}$. Thus, the partial order structure on the $U_{n,i}$'s is the same as that on the $O_{n,i}$'s.

Recall the partial order $\prec$ on $T(X)$. We can use $\prec$ to describe $U_{n,i}$. We get that $U_{n,i} = \{x \in T(X) | v_{n,i} \prec x\}$ (where $v_{n,i}$ is as in the construction of $T(X)$).

Just as the sets $\{O_{n,i}\}$ form a base for the "neighborhood of ∞" in X, the sets $\{U_{n,i}\}$ form a base for the "neighborhood of ∞" in $T(X)$.

We now prove that $T(X)$ has the (α, n) complementation property.

Proposition 49 *If X has the (α, n) complementation property, then so does $T(X)$.*

Proof: What we do first is to show that $cl(T(X)) \subset I\!R^2$ is the maximal totally disconnected compactification of $T(X)$. To do this, we use Proposition 3.12 of [MC1] (Lemma 34). Clearly $R(X)$ is countable (since it is the image of the countable set $\alpha X \setminus X$ under the map f^α). Thus, $R(X)$ is totally disconnected. Let $U \subset cl(T(X))$ be an open connected neighborhood of a point $r \in R(X)$. We wish to show that $U \setminus R(X)$ is connected. Suppose that $U \setminus R(X) = U \cap T(X)$ is not connected. Let A_1 and A_2 be two distinct components of $U \cap T(X)$. Then $C_i = cl_{I\!R^2}(A_i)$ is connected. We can modify C_1 slightly so that $C_1 \cap C_2 = \emptyset$ and $cl(C_1) \cap R(X) = cl(A_1) \cap R(X) \neq \emptyset$ (one way to see this is that since $C_1 \cap R(X) \neq \emptyset$, C_1 cannot be compact. Thus, for large enough n, $cl_{I\!R^2}(C_1 \setminus T_n(X)) \cap R(X) = cl_{I\!R^2}(C_1) \cap R(X)$). However, then by Lemma 45, $cl_{I\!R^2}(C_1) \cap cl_{I\!R^2}(C_2) = \emptyset$. But $cl(C_i) \cap R(X) = cl(A_i) \cap R(X)$ and $cl(A_1) \cap cl(A_2) \neq \emptyset$ since U is connected. Thus, $U \cap T(X) = U \setminus R(X)$ is connected so $cl(T(X))$ is the maximum totally disconnected compactification of $T(X)$ by Proposition 3.12 in [MC1].

The map f^α is a bijection when restricted to $\alpha X \setminus X$ (this is a consequence of Lemmas 42 and 43). Thus, it is a homeomorphism and R has type (α, n) if and only if $\alpha X \setminus X$ has type (α, n) if and only if X has the (α, n) complementation property. Thus, the maximum countable compactification of $T(X)$ is of type (α, n) so by Theorem 35, $T(X)$ has the (α, n) complementation property. ∎

We can use the same ideas as in the proof of this proposition to prove that $cl(U_{n,i})$ has the (σ, m) complementation property if $cl(O_{n,i})$ has the (σ, m) complementation property.

3.5 Construction of the map $g : T(X) \to T(Y)$

We will need some further properties of $T(X)$ in order to construct the map $g : T(X) \to T(Y)$. Recall our convention that X and Y have the (α, n) complementation property.

Lemma 50 *Let $n = 1$ and $\sigma < \alpha$ and $k \in I\!N$. Then there is some $m \in I\!N$ so that*

$$T(X) \setminus T_m(X) = T \cup (\bigcup_{i=1}^{k} S_i) \cup (\bigcup_{i=1}^{l} R_i)$$

with $\Psi(T) = \alpha$ and $\Psi(S_i) \geq \sigma$ and $\Psi(R_i) < \alpha$.

Proof: The idea is to translate properties of $T(x_{n,i}, X)$ into properties of $U_{n,i}$ into properties of $O_{n,i}$ and then into properties of $\alpha X \setminus X$ which is homeomorphic to $W(\omega^\alpha + 1)$ (here ω is the first infinite ordinal). We illustrate this in the below.

$$T(x_{n,i}, X) \Longleftrightarrow U_{n,i} \Longleftrightarrow O_{n,i} \Longleftrightarrow \alpha X \setminus X \cong W(\omega^\alpha + 1)$$

However, we already have the desired decomposition of $\alpha X \setminus X$ as given in Proposition 39. Let $\hat{S}_i$ and $\hat{T}$ and $\hat{R}_i$ be the decomposition from Proposition 39. Then using the map $g : X \to T(X)$ we get a decomposition S_i and T and R_i. However, this will not cover $T_m(X)$, so we need to include this in our decomposition. As we mention at the end of the last section, $cl(U_{n,i})$ has the (σ, l) complementation property if and only if $cl(U_{n,i})$ has the (σ, l) complementation property. Thus we know that $\Psi(S_i) \geq \sigma$ and $\Psi(T) = \alpha$ and $\Psi(R_i) < \alpha$. ∎

Lemma 51 *If $n > 1$, then there is some $m \in I\!N$ so that*

$$T(X) \setminus T_m(X) = (\bigcup_{i=1}^{n} S_i) \cup (\bigcup_{i=1}^{k} R_i)$$

where $\Psi(S_i) = \alpha$ and $\Psi(R_i) < \alpha$.

Proof: The proof of this lemma follows from Proposition 41 like the preceeding lemma followed from Proposition 39. ∎

These two lemmas will be used in the inductive construction of the map.

Now, we discuss what we call the "trash procedure".

Trash Procedure

The fact that we are not trying to get an injective map makes things much easier. At each stage of our inductive construction, we only need a surjective map. So, if we have already defined a perfect surjective map $g : A \subset T(X) \to T(Y)$, we have some freedom in defining g on $T(X) \setminus A$. What we will always do is choose this so as to guarantee that the map is continuous and perfect. Clearly we can map any $T(X)$ onto $[0, 1)$ in a perfect way (just map $T_n(X)$ onto $[0, 1 - \frac{1}{n})$). Thus, if have a map $g : A \to T(X)$ (as above) we take the left-over branches and choose *any* single ray in $T(X)$ to map $T(X) \setminus A$ onto. This will be a perfect map.

The intuitive idea is that we want to use $T(X)$ to "cover" $T(Y)$. If we can "cover" $T(Y)$ using a sub-tree of $T(X)$, then it does not matter that much what we do with the rest of $T(X)$ - as long as the resulting map is perfect.

Now we proceed with the construction of the map $g : T(X) \to T(Y)$.

The following figure will help in understanding the constructions.

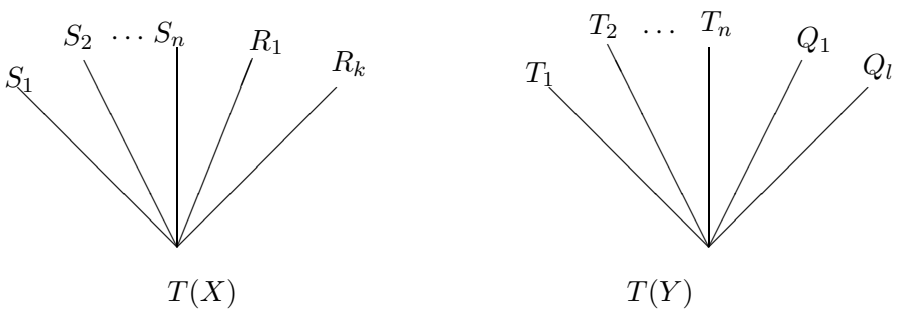

Figure 6: The map $g : T(X) \to T(Y)$

Theorem 5.3 in [CChF] and Theorem 2 in [ChT] are the cases where $(\alpha, n) = (0, n)$.

Thus, suppose that it is true for all $(\sigma, m) < (\alpha, n)$. There are two cases in the induction step.

Case 1: Going from $(\alpha, 1)$ to (α, n).

By lemma 51 there is some $m \in I\!N$ so that

$$T(Y) \setminus T_m(Y) = (\bigcup_{i=1}^{n} T_i) \cup (\bigcup_{i=1}^{l} Q_i)$$

where the $cl(T_i)$'s are pairwise disjoint and $cl(T_i)$ is a dendrite with $\Psi(cl(T_i)) = \alpha$ and the $cl(Q_i)$'s are pairwise disjoint (and disjoint from all the T_i's) with $cl(Q_i) < \alpha$. Notice that T_i has the $(\alpha, 1)$ complementation property. Let the endpoint of $T_m(Y)$ corresponding to T_i be y_i. Now, again by lemma 51, there is some $p \in I\!N$ so that

$$T(X) \setminus T_p(X) = (\bigcup_{i=1}^{n} S_i) \cup (\bigcup_{i=1}^{k} R_i)$$

with the corresponding properties. Let x_i be the endpoint of $T_p(X)$ corresponding to S_i.

For $i = 1, \ldots, n-1$ we map the arc from r_x to x_i onto the arc from r_y to y_i. Then we use the induction hypothesis to map S_i perfectly onto T_i. We map the arc from r_x to x_n to r_y. By the induction hypothesis, we can map S_n perfectly onto $T(Y) \setminus (\bigcup_{i=1}^{n-1} T_i \cup \text{arc from } \mathrm{r_y} \text{ to } \mathrm{y_i})$. Then use the "trash procedure" on the R_i's.

Case 2: Going from (σ, m) for all $\sigma < \alpha$ and $m \in I\!N$ to $(\alpha, 1)$. This includes the case where α is a successor ordinal.

We do this by inducing on the "levels" in $T(Y)$. We know that

$$T(Y) = T_1(Y) \cup \bigcup_{i=1}^{|I_1|} T(y_{1,i}, Y)$$

With no loss of generality, we assume that $\Psi(T(y_{1,1}, Y)) = \alpha$ and $\Psi(T(y_{1,i}, Y)) < \alpha$ for $i = 2, 3, \ldots, |I_1|$. By Proposition 29, there is some $\sigma < \alpha$ so that $\Psi(T(y_{1,i}, Y)) < \sigma$ for $i = 2, 3, \ldots, |I_1|$. Thus by Lemma 50 there is some $p \in I\!N$ so that

$$T(X) \setminus T_p(X) = T \cup (\bigcup_{i=2}^{|I_1|} S_i) \cup (\bigcup_{i=1}^{l} R_i)$$

so that $\Psi(T) = \alpha$ and $\Psi(S_i) \geq \sigma$. We map $T_p(X)$ onto r_y and by the induction hypothesis we can map S_i perfectly onto $T(y_{1,i}, Y)$ for $i = 2, 3, \ldots, |I_1|$. Finally, we use the "trash procedure" on R_i.

Now, we just repeat the above procedure using $T \subset T(X)$ instead of $T(X)$ and $T(y_{1,1}, Y)$ instead of $T(Y)$.

Comments

The best way to understand the construction for Case 2 is to consider the case of the $(1,1)$ complementation property. Suppose that $T(X)$ is as "simple" as possible (*i.e.* that each branching point in $T(X)$ has only two branches one of which has type $(0,1)$ and the other of which has type $(1,1)$). Then if $T(Y)$ is more complicated, you need to "climb up" $T(X)$ sufficiently far in order to get enough branches to "cover" the initial segments of $T(Y)$.

Here is an illustration of the above situation.

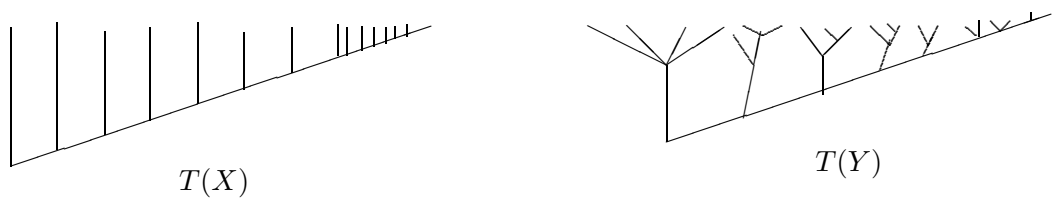

Figure 7: Example for $T(X) \to T(Y)$

3.6 Construction of the map $h : T(Y) \to Y$

Because of the natural way we constructed $T(Y)$ (using the structure of Y), this construction is rather natural and is not that difficult once we see the basic idea.

The basic tool in this construction is the fact that any Peano space is the image of [0,1]. Since $T(Y)$ is constructed from many copies of [0,1], we use this fact to get the desired map.

The cases of the $(0,n)$ complementation property are proved in [CChF] Theorem 5.4.

Theorem 52 *([CChF], Theorem 5.4) If Y is a locally connected generalized continuum with the $(0,n)$ complementation property, then there is a perfect surjection $F : T_n(1) \to Y$.*

Once we have the general setup of the nested collection of open sets $\{\mathcal{O}_m\}$ and the spaces $T(X)$, our proof is a rather straightforward adaptation of the proof of Theorem 52 given in

[CChF]. Like that proof, we make heavy use of the following lemma (which is Lemma 5.1 in [CChF], modified slightly).

Lemma 53 *Suppose that Z is a locally connected continuum, $U \subset Z$ is an open connected subset, and $C \subset U$ is compact. Then there is a locally connected continuum K such that $C \subset K \subset U$.*

We now start the construction of h.

Induction

$n = 0$:

$T_1(Y) = \mathcal{T}_{c(0)}(1, (0, 1))$

and $c(0)$ is the number of components of $\mathcal{O}_1$. We know that $Y \setminus \mathcal{O}_2$ is a compact subset of Y which is an open connected set. So by Lemma 53, there is a Peano space Z with $Y \setminus \mathcal{O}_2 \subset Z \subset Y$. Now Z is compact, so there is an $M(0) \in I\!N$ so that $Z \cap \mathcal{O}_k = \emptyset$ if $k \geq M(0)$. Choose $y_{1,j} \in bd(O_{1,j})$ for all $j \in I_1$. Also choose any point $y_0 \in Y \setminus \mathcal{O}_1$. We map $T_1(Y)$ onto Z in such a way as to map the root onto y_0 and the vertex $v_{1,i}$ onto $x_{1,i}$.

One way to do this is to map the first 1/2 of the first branch of $T_1(Y)$ onto Z and then use the second 1/2 to get an arc to $x_{1,1}$ from wherever the point half-way up the first branch is mapped to. Then we do the same for all of the other branches. Notice that this map is certainly not going to be injective, but it will be surjective onto $Y \setminus \mathcal{O}_1$.

Step $n + 1$:

$T_{n+1}(Y) = T_n(Y) \cup (\bigcup_{i \in I_n} T_{n+1,i}(Y)$ where for all $i \in I_n$,

Again, $cl(O_{n,i}) \setminus \mathcal{O}_{n+1}$ is a compact subset of $O_{n-1,a(n,i)}$ which is a connected open set. Thus by Lemma 53 there is a Peano space Z with $cl(O_{n,i}) \setminus \mathcal{O}_{n+1} \subset Z \subset O_{n-1,a(n,i)}$. Furthermore, there is an $M(n, i) \in I\!N$ so that if $k \geq M(n, i)$ then $\mathcal{O}_k \cap Z = \emptyset$. Now choose $x_{n+1,j} \in bd(O_{n+1,j})$ for all $j \in s(n, i)$. We map $T_{n+1,i}$ onto Z in such a way as to map the root of $T_{n+1,i}$ to $x_{n-1,i}$ and the vertex $v_{n+1,j}$ to $x_{n+1,j}$.

Again, one way to do this is to map the first 1/2 of the first branch of $T_{n+1,i}$ onto Z

and then use the second 1/2 of the branch to get an arc from wherever 1/2 was mapped to to the point $x_{n+1,j}$. We do the same for all of the other branches of $T_{n+1,i}$.

The important thing is that we map $T_{n+1,i}$ onto Z which contains $cl(O_{n,i}) \setminus \mathcal{O}_{n+1}$ and we map $T_{n+1,i}$ into $O_{n-1,a(n,i)} \setminus \mathcal{O}_{M(n,i)}$, thus keeping the image a positive distance away from $\alpha Y \setminus Y$ (this is what makes the map perfect).

End Induction

With this definition, h is continuous since each of the component maps are and they agree on their common domains (the vertices). Furthermore, h is surjective since each of the component maps is.

The only thing left to prove is that h is a perfect map. We do this now.

Proposition 54 *The map h constructed above is a perfect map.*

Proof: Let $K \subset Y$ be compact. Then there is an $n \in I\!N$ so that $K \subset \mathcal{O}_n \setminus cl(\mathcal{O}_{n+1})$. Thus $h^{-1}(K) \subset cl(T_{n+1}(Y))$ is a closed subset of a compact set so is compact.

Let $C \subset T(Y)$ be closed. Then $C \cap cl(T_n(Y))$ is compact for every $n \in I\!N$. Let $y_k = h(t_k)$ where $t_k \in C$ and $y_k \to y$. Since $Y = \bigcup_{n \geq 0}(Y \setminus \mathcal{O}_n)$, there is some $n \in I\!N$ with $y \in Y \setminus cl(\mathcal{O}_n)$. Then there is an $m \in I\!N$ so that if $k \geq m$ then $h(t_k) = y_k \notin \mathcal{O}_n$ so $t_k \in T_{n+1}(Y)$ for all $k \geq m$. This implies that $t_k \in C \cap T_{n+1}(Y)$ for all $k \geq m$. Since $C \cap cl(T_{n+1}(Y))$ is compact, there is some $t \in C \cap cl(T_{n+1}(Y))$ which is a cluster point of the t_k's. However, then $h(t_k)$ clusters at $h(t)$ but $h(t_k)$ converges to y. Thus, $h(t) = y$ or $h(C)$ is closed. ∎

3.7 Consequences and Discussion

What we have proven is that if X and Y are two locally connected generalized continua both with the (α, n) complementation property, then there are perfect surjections $f : X \to Y$ and $g : Y \to X$. The fact that f and g are perfect implies that we can extend these maps to $f^\beta : \beta X \to \beta Y$ and $g^\beta : \beta Y \to \beta X$. Furthermore, $(f^\beta)^{-1}(\beta Y \setminus Y) = \beta X \setminus X$ and similarly

for g^β. Thus, f^β and g^β preserve the remainder of the Stone-Čech compactifications of X and Y. So, we get the following corollary to Theorem 36.

(What we have really shown is that we can extend these maps to $f^\alpha : \alpha X \to \alpha Y$ and $g^\alpha : \alpha Y \to \alpha X$ with $f^{-1}(\alpha Y \setminus Y) = \alpha X \setminus X$ and similarly for g. This is a much stronger statement than the one about f^β and g^β.)

Corollary 55 *If X and Y are locally connected generalized continua both with the (α, n) complementation property, then the set of remainders of X and Y are the same.*

Proof: By Theorem 2.1 from [MG1], we know that any image of a remainder is a remainder (this is also Theorem 20 in the background chapter). Let K be a remainder of X. Then the canonical projection $\pi : \beta X \setminus X \to K$ composed with g^β gives a map $(g^\beta \circ \pi) : \beta Y \setminus Y \to K$, so that K is an image of a remainder of Y. Thus, every remainder of X is also a remainder of Y. The same argument with f instead of g shows that every remainder of Y is also a remainder of X.∎

However, this does not mean that $K(X)$ is isomorphic to $K(Y)$. Here is an easy example to illustrate this.

Example Let $X = [0, \infty)$ and $Y = \mathbb{R}^2$. Clearly both X and Y are locally connected generalized continua. Also clearly both X and Y have the $(0, 1)$ complementation property. Thus, the set of remainders of X is the same as the set of remainders of Y. However, $\beta X \setminus X$ is an indecomposable continuum whereas $\beta Y \setminus Y$ is a decomposable continuum [C1]. By a theorem of Magill [MG2], we know that $K(X)$ is isomorphic to $K(Y)$ if and only if $\beta X \setminus X$ is homeomorphic to $\beta Y \setminus Y$ (much more about this in the next chapter.)

So, we have a condition as to when the set of remainders of two locally compact generalized continua are the same. How about conditions on when the lattices of Hausdorff compactifications are the same? This is a much harder question, and one that we will discuss in the next chapter.

Now let us compare the ideas in this chapter to those in [CChF]. Our Theorem 36 is clearly meant to be an extension of Theorem 5.3 and Theorem 5.4 of [CChF]. There are several additional difficulties in the countable case. The first difficulty is what to use in place of T_n? Clearly, there is only one such choice for the finite case. However, for the countable case there are many different choices (as we saw, $T(X)$ can look very different for different X's). What solves this problem is the choice of the open sets $\{\mathcal{O}_n\}$. Using this collection as a guide, it is easy to get the desired replacement for T_n. However, this raises a second problem – there are many of these spaces so we need to be able to get a perfect surjection between any two of them. Once we see how to do this, the rest is modeled on the proofs in [CChF].

Chapter 4

Lattice of Compactifications

4.1 Introduction

It is of interest to ask when $K(X)$ is isomorphic to $K(Y)$ for Tychonoff spaces X and Y. We saw in the background chapter (Chapter 2) that $K(W(\omega_\alpha))$ is trivial if $\alpha \geq 1$. Thus, $K(W(\omega_\alpha)) \cong K(W(\omega_\delta))$ if $\alpha \geq 1$ and $\delta \geq 1$. This is an interesting example. Notice that $W(\omega_\alpha)$ is not homeomorphic to $W(\omega_\delta)$ if $\alpha \neq \delta$ (this is obvious, since they do not have the same cardinality). Thus, this is an infinite family of non-homeomorphic spaces which have the same lattice of Hausdorff compactifications.

In this chapter (until section 4.4) all spaces will be locally compact and Tychonoff, by which we mean completely regular and Hausdorff. The main results in this chapter are Theorems 66 and 70. The other theorems are basically refinements of these results.

Under what conditions is $K(X) \cong K(Y)$? This will be the main question under investigation for this chapter.

Here is an easy result which gives a sufficient condition for $K(X) \cong K(Y)$.

Proposition 56 *Let X and Y be spaces. Suppose that there are compact sets $X_0 \subset X$ and $Y_0 \subset Y$ so that $X \setminus int(X_0)$ is homeomorphic to $Y \setminus int(Y_0)$, then $K(X) \cong K(Y)$.*

Proof: What we do is construct a function $\Gamma : K(X) \to K(X \setminus X_0)$. Clearly if we can show that Γ is an isomorphism, the proof for Y and $Y \setminus Y_0$ will be the same. Furthermore, it is clear that $K(X \setminus X_0) \cong K(Y \setminus Y_0)$.

Let $\alpha X \in K(X)$. Then since $X \setminus int(X_0) \subset X \subset \alpha X$, we know that $cl(X \setminus int(X_0)) \subset \alpha X$ is a compactification of $X \setminus int(X_0)$. We let $\Gamma(\alpha X)$ be this compactification. Clearly Γ as

defined is injective. To see that it is surjective, we take $\delta(X \backslash int(X_0)) \in K(X \backslash int(X_0))$. Let $\hat{\delta}X$ be $X_0 \coprod \delta(X \backslash int(X_0))$ (topological sum) with the obvious identification of the boundary of X_0 (as a subset of X) with the boundary of X_0 viewed as a subset of $\delta(X \setminus int(X_0))$. Then X can be thought of as a subset of $X_0 \cup \delta(X \setminus int(X_0))$, so that $\hat{\delta}X$ can be thought of as a compactification of X (since X is dense in $\hat{\delta}X$). Then it is easy to check that $\Gamma(\hat{\delta}X) = \delta(X \setminus int(X_0))$. Thus, Γ is surjective.

To see that Γ preserves the order, just note that if $\alpha X \leq \delta X \in K(X)$, then there is an $f : \delta X \to \alpha X$ so that $f|_X = id_X$. Thus, if we let $\hat{f} = f|_{cl(X \backslash int(X_0))}$ (where the closure is in δX), then it is easy to see that $\hat{f} : \Gamma(\delta) \to \Gamma(\alpha)$ with $\hat{f}|_{X \backslash int(X_0)} = id_{X \backslash int(X_0)}$. Thus, $\Gamma(\alpha X) \leq \Gamma(\delta X)$, so Γ preserves order.

Using a similar procedure we can show that Γ^{-1} also preserves order. (Notice that we essentially constructed Γ^{-1} when we proved that Γ was surjective, above). Thus, Γ is an isomorphism. ∎

This is by no means the whole story, however. The example of $W(\omega_\alpha)$ shows us this ($W(\omega_\alpha)$ was defined in the Chapter 2, the background chapter). Let $\alpha \neq \delta$ be ordinals. Then if C is any compact set in $W(\omega_\alpha)$, we know $W(\omega_\alpha) \backslash C$ has cardinality equal to the cardinality of ω_α. Thus there is no way that $W(\omega_\alpha) \backslash C$ and $W(\omega_\delta) \backslash D$ could be homeomorphic. However, as we mentioned in the background chapter, $K(W(\omega_\alpha)) \cong K(W(\omega_\delta))$ provided that both α and δ are greater than 1.

Thus, we need to look for other conditions. Magill provided such a condition in his paper [MG2]. Theorem 12 and Theorem 13 from his paper imply the following theorem.

Theorem 57 *(Magill,[MG2]) Let X and Y be two locally compact Tychonoff spaces. Let $K(X)$ ($K(Y)$) be the lattice of Hausdorff compactifications of X (Y) ordered in the usual way, and let βX (βY) be the Stone-Čech compactification of X (Y). Then $K(X)$ is isomorphic to $K(Y)$ if and only if $\beta X \setminus X$ is homeomorphic to $\beta Y \setminus Y$.*

What we do in this chapter is to provide a new proof of this result using function algebra techniques. This brings new tools to the subject which shed new light on the situation. We

also prove several extensions of Magill's result, most of them with almost trivial proofs using the new techniques. Using algebraic techniques, many of the necessary manipulations can be handled naturally by the algebra.

4.2 Preliminaries

Notational Conventions

If $\mathcal{I}$ and $\mathcal{J}$ are two (closed) subalgebras of a function algebra $\mathcal{A}$, we denote by $\mathcal{I} \oplus \mathcal{J}$ the (closed) subalgebra of $\mathcal{A}$ generated by $\mathcal{I}$ and $\mathcal{J}$. As a special case, we denote by $\mathcal{I} \oplus 1$ the algebra generated by $\mathcal{I}$ and the constant functions.

The algebraic sum of $\mathcal{A}$ and the constant functions is closed if and only if $\mathcal{I}$ is closed, taking the closure is unnecessary in this case.

Further, if $\{\mathcal{I}_\alpha\}$ is a collection of elements in some partially ordered set $\mathcal{P}$, we denote by $\bigvee\{\mathcal{I}_\alpha\}$ the smallest element of $\mathcal{P}$ larger than all of the $\mathcal{I}_\alpha$'s, when it exists. Similarly, we denote by $\bigwedge\{\mathcal{I}_\alpha\}$ the largest element of $\mathcal{P}$ smaller than all of the $\mathcal{I}_\alpha$'s, when it exists.

We let $\mathbf{Z}(\mathcal{I})$ be the maximal zero set of the ideal $\mathcal{I}$, *i.e.* $\mathbf{Z}(\mathcal{I}) = S$ if $\mathcal{I} = \{f \in C^*(X) | f|_S = 0\}$. For a collection of functions $\mathcal{F}$, H is a *stationary set* if $f|_H$ is constant for every $f \in \mathcal{F}$. We will usually be interested in the *maximal* stationary sets.

Now we will prove a couple of easy results which will prove useful to us in the sequel. The second one is just the Chinese Remainder Theorem ([La], page 63) for $C^*(H)$ for compact Hausdorff H.

Proposition 58 *Let H be a compact Hausdorff space and let $\mathcal{I} \subset C^*(H)$ be a closed ideal. Let S be the zero set of $\mathcal{I}$ so that $\mathcal{I} = \{f \in C^*(H) |\ f|_S = 0\}$. Then $C^*(H)/\mathcal{I} \cong C^*(S)$.*

Proof: We define $\Upsilon : C^*(H) \to C^*(S)$ by $f \mapsto f|_S$. Notice that Υ is an algebra homomorphism since restriction is an algebra homomorphism. Also Υ is surjective since S is a compact subset of H and therefore C^*-embedded in H. All we have left to show is injectivity. Obviously, Υ is not injective. However, $\ker(\Upsilon) = \mathcal{I}$, since $\mathcal{I}$ is exactly the set of functions which are 0 on S. Thus Υ induces an isomorphism between $C^*(H)/\mathcal{I}$ and

$C^*(S)$. Finally, Υ preserves the norm since it is an algebra isomorphism and the norms in both $C^*(H)$ and $C^*(S)$ are determined by the algebraic structure ([GJ], pp. 12-13). Here we put the supremum norm on $C^*(S)$ and the quotient norm on $C^*(H)/\mathcal{I}$. ∎

Lemma 59 (Chinese Remainder Theorem)
Let $\mathcal{I}_i$ be ideals in $C^*(H)$ for $i = 1, \ldots, n$. Suppose that $\mathcal{I}_i \bigvee \mathcal{I}_j = C^*(H)$ for all $i \neq j$. Let $\mathcal{I} = \bigcap \mathcal{I}_i$. Then $C^*(H)/\mathcal{I} \cong C^*(H)/\mathcal{I}_1 \oplus \cdots \oplus C^*(H)/\mathcal{I}_n$.

Proof: Let $S_i \subset H$ be the zero set of $\mathcal{I}_i$ so that $\mathcal{I}_i = \{f \in C^*(H) |\ f|_{S_i} = 0\}$. Then the conditions on the $\mathcal{I}_i$'s insure that $S_i \cap S_j = \emptyset$ for $i \neq j$. Let $S = \cup S_i$ so that $\mathcal{I} = \{f \in C^*(H) |\ f|_S = 0\}$. By Proposition 58, $C^*(H)/\mathcal{I} \cong C^*(S)$ and $C^*(H)/\mathcal{I}_i \cong C^*(S_i)$. Since $S_i \cap S_j = \emptyset$ for $i \neq j$, we know that $C^*(S) = C^*(S_1) \oplus \cdots \oplus C^*(S_n)$. ∎

The only difference between this result and the usual Chinese Remainder Theorem is that we have a norm on $C^*(X)$. The algebraic result here is no different from the usual Chinese Remainder Theorem, but we need to prove the isomorphism for C^*-algebras.

4.3 Main Results

The main result of this section is Theorem 66 which gives an algebraic condition on $C^*(X)$ and $C^*(Y)$ for when $K(X)$ is isomorphic to $K(Y)$. Using this, we then get some results as to when an interval in $K(X)$ can be embedded as an interval of $K(Y)$. We need to prove some lemmas first. These lemmas relate $K(X)$ with $C^*(\beta X \setminus X)$.

Lemma 60 $C^*(X)/C_0(X) \cong C^*(\beta X \setminus X)$.

Proof: $C_0(X)$ is an ideal in $C^*(X)$ and $C_0(X) = \{f \in C^*(X) |\ f^\beta|_{\beta X \setminus X} = 0\}$ where f^β is the extension of f to βX. Now the lemma follows from Proposition 58 using $\mathcal{I} = C_0(X)$ and $H = \beta X$. ∎

Notation 12 *$PSA(X)$ = lattice of all closed unital subalgebras of $C^*(X)$ which separate points from closed sets.*

By the remarks above, $PSA(X)$ is lattice isomorphic to $K(X)$.

Notation 13 *$CA(X)$ = lattice of all closed unital subalgebras of $C^*(X)$.*

$NI(X)$ = the partially ordered set of all closed non-maximal ideals in $C^*(X)$.

$MI(X)$ = the collection of all maximal ideals in $C^*(X)$

Lemma 61 $PSA(X) \cong CA(\beta X \setminus X)$

Proof: Letting Υ be the isomorphism from Lemma 60, we define a function $\Upsilon^{\#} : PSA(X) \to CA(\beta X \setminus X)$ by $\mathcal{A} \mapsto \{\Upsilon(f) : f \in \mathcal{A}\}$. Now, from the correspondence theorem ([RO], p. 26) from elementary algebra, $\Upsilon^{\#}$ is an order preserving bijection between the set of subalgebras of $C^*(X)$ which contain $\ker(\Upsilon)$ and the subalgebras of $C^*(X)$. However, the set of subalgebras of $C^*(X)$ which contain $\ker(\Upsilon)$ is exactly $PSA(X)$, as discussed above. ∎

Now we prove a series of results with the goal of proving Theorem 66. The propositions show the major steps in the proof of this theorem. Theorem 65 is of independent interest since it indicates that the lattice structure of $CA(X)$ is sufficient to determine $C^*(X)$.

Proposition 62 *Let X and Y be compact Hausdorff spaces. Suppose that there is a lattice isomorphism $\phi : CA(X) \to CA(Y)$. Then there is an isomorphism of partially ordered sets $\psi : NI(X) \to NI(Y)$.*

Proof: We show that $\mathcal{A} = \mathcal{I} \oplus 1$ for some ideal $\mathcal{I}$ if and only if $[\mathcal{A}, C^*(X)] \cong CA(S)$ for some compact Hausdorff S. Here $[\mathcal{A}, C^*(X)] \equiv \{\mathcal{B} \in CA(X) | \mathcal{A} \leq \mathcal{B} \leq C^*(X)\}$.

First suppose that $\mathcal{A} = \mathcal{I} \oplus 1$, then by Proposition 58 $C^*(X)/\mathcal{I} \cong C^*(S)$, where $S = Z(\mathcal{I})$. Now this isomorphism takes $\mathcal{B} \in [\mathcal{A}, C^*(X)]$ to some closed unital algebra in $CA(S)$. Furthermore, this correspondence between elements in $[\mathcal{A}, C^*(X)]$ and closed unital algebras in $C^*(S)$ is order preserving and bijective. Thus, $CA(S) \cong [\mathcal{A}, C^*(X)]$.

Conversely, suppose that $[\mathcal{A}, C^*(X)] \cong CA(S)$ for some compact Hausdorff S. Denote this isomorphism by Σ. We wish to show that $\mathcal{A} = \mathcal{I} \oplus 1$ for some ideal $\mathcal{I}$. Suppose not. Then there are two disjoint maximal stationary sets S_1 and S_2 of $\mathcal{A}$. Let C_1 and C_2 be dual points of $CA(X)$ so that $\mathcal{I}_1 \oplus 1 \leq C_1$ and $\mathcal{I}_2 \oplus 1 \leq C_2$ where $\mathcal{I}_1$ and $\mathcal{I}_2$ are the two ideals with $Z(\mathcal{I}_1) = S_1$ and $Z(\mathcal{I}_2) = S_2$.

We need to discuss some properties of dual points in the lattice $CA(X)$. The closed unital algebra $\mathcal{C} \in CA(X)$ is a dual point if $\mathcal{C} \neq C^*(X)$ and if $\mathcal{B} > \mathcal{C}$ then $\mathcal{B} = C^*(X)$.

Dual points in $CA(X)$ correspond to quotients H where there are only two points in X which are identified under the quotient $\pi : X \to H$.

For two distinct dual points D_1 and D_2 in $CA(X)$, there are only two possibilities. Let $D_1 \sim \{a, b\}$ and $D_2 \sim \{c, d\}$ (where by $\sim$ we mean under the correspondence of $CA(X)$ and $K(X)$).

Case 1: $D_1 \wedge D_2$ has two dual points larger than it.

In this case, it is easy to see that $\{a, b\} \cap \{c, d\} = \emptyset$

Case 2: $D_1 \wedge D_2$ has three dual points larger than it.

In this case, $\{a, b\} \cap \{c, d\} \neq \emptyset$.

(For our dual points C_1 and C_2, we are in case 1, since $S_1 \cap S_2 = \emptyset$).

Now let D_1, D_2, and D_3 be three dual points in $CA(X)$ with D_1 and D_2 in case 1 from above. There are three possibilities. Let $D_1 \sim \{a, b\}$ and $D_2 \sim \{c, d\}$ and $D_3 \sim \{e, f\}$.

Case 1a: $D_1 \wedge D_2 \wedge D_3$ has 6 dual points larger than it.

This case corresponds to $\{a, b\} \cap \{e, f\} \neq \emptyset$ and $\{c, d\} \cap \{e, f\} \neq \emptyset$.

Case 1b: $D_1 \wedge D_2 \wedge D_3$ has 3 dual points larger than it.

This case corresponds to $\{a, b\} \cap \{e, f\} = \emptyset$ and $\{c, d\} \cap \{e, f\} = \emptyset$.

Case 1c: $D_1 \wedge D_2 \wedge D_3$ has 4 dual points larger than it.

This case corresponds to $\{a, b\} \cap \{e, f\} = \emptyset$ and $\{c, d\} \cap \{e, f\} \neq \emptyset$.

There is some dual point $C_3 \in CA(S)$ so that $\Sigma(C_1) \wedge \Sigma(C_2) \wedge C_3$ is in case 1a. Then since Σ is an isomorphism, $C_1 \wedge C_2 \wedge \Sigma^{-1}(C_3)$ is in case 1a in $[\mathcal{A}, C^*(X)]$. However, this is impossible by the choice of C_1 and C_2 (they lie above disjoint maximal stationary sets

of $\mathcal{A}$). Thus $\mathcal{A}$ cannot have two disjoint maximal stationary sets if $[\mathcal{A}, C^*(X)] \cong CA(S)$. However, this means that $\mathcal{A} = \mathcal{I} \oplus 1$.

So, now we can identify elements of the form $\mathcal{I} \oplus 1$ in $CA(X)$. Choose $\mathcal{I} \in NI(X)$. Then $\mathcal{I} \oplus 1 \in CA(X)$ so $\phi(\mathcal{I} \oplus 1) \in CA(Y)$. Furthermore, $\phi(\mathcal{I} \oplus 1) = \mathcal{J} \oplus 1$ for some $\mathcal{J} \in NI(Y)$ since ϕ is an isomorphism so $CA(S) \cong [\mathcal{I} \oplus 1, C^*(X)] \cong [\mathcal{J} \oplus 1, C^*(Y)]$. Thus define $\psi(\mathcal{I}) = \mathcal{J}$.

Let $\mathcal{I}_1 \leq \mathcal{I}_2 \in NI(X)$. Then $\mathcal{I}_1 \oplus 1 \leq \mathcal{I}_2 \oplus 1$ in $CA(X)$ so $\mathcal{J}_1 \oplus 1 = \phi(\mathcal{I}_1 \oplus 1) \leq \phi(\mathcal{I}_2 \oplus 1) = \mathcal{J}_2 \oplus 1$ in $CA(Y)$. This means that $\mathcal{J}_1 \leq \mathcal{J}_2$. Thus ψ is order-preserving. It is injective since ϕ is injective so we know that ψ is an isomorphism. ∎

Proposition 63 *An isomorphism of partially ordered sets $\psi : NI(X) \to NI(Y)$ induces a bijection $F_\psi : MI(X) \to MI(Y)$.*

Proof: Let $M_x \in MI(X)$, and let $\{\mathcal{I}_\alpha\}$ be an (infinite) increasing net in $NI(X)$ so that $M_x = \bigvee_\alpha \mathcal{I}_\alpha$. Define $F_\psi(M_x) = \bigvee_\alpha \psi(\mathcal{I}_\alpha)$. First we show that this is well defined. Thus let $\{\mathcal{I}_\alpha\}_{\alpha\in\Lambda}$ and $\{\mathcal{J}_\gamma\}_{\gamma\in\Xi}$ be increasing nets in $NI(X)$ so that $\bigvee_\alpha \mathcal{I}_\alpha = M_x = \bigvee_\gamma \mathcal{J}_\gamma$. Let $\Omega = \Lambda \times \Xi$ and define $(\lambda_1, \gamma_1) \leq (\lambda_2, \gamma_2)$ if $\lambda_1 \leq \lambda_2$ and $\gamma_1 \leq \gamma_2$. It is easy to see that with this definition Ω is a directed set. Let $\mathcal{K}_{(\lambda,\gamma)} = \mathcal{I}_\lambda \wedge \mathcal{J}_\gamma \in NI(X)$. If $\bigvee_\omega \mathcal{K}_\omega \in NI(X)$ then there is some $y \neq x$ so that $y \in Z(\bigvee_\omega \mathcal{K}_\omega) = \bigcap_\omega Z(\mathcal{K}_\omega) = \bigcap_{(\lambda,\gamma)}(Z(\mathcal{I}_\lambda) \cup Z(\mathcal{J}_\gamma))$ so that for all $(\lambda, \gamma) \in \Omega$ we have $y \in Z(\mathcal{I}_\lambda)$ or $y \in Z(\mathcal{J}_\gamma)$. However, since $\bigvee_\lambda \mathcal{I}_\lambda = M_x$ there is an $\lambda_0 \in \Lambda$ so that if $\lambda \geq \lambda_0$ then $y \notin Z(\mathcal{I}_\lambda)$. Similarly, $y \notin Z(\mathcal{J}_\gamma)$ for $\gamma \geq \gamma_0$. Thus, $\bigvee_\omega \mathcal{K}_\omega \notin NI(X)$. Clearly $\bigvee_\omega \mathcal{K}_\omega \leq M_x$ so $\bigvee_\omega \mathcal{K}_\omega = M_x$. Let $N = \bigvee_\lambda \psi(\mathcal{I}_\lambda)$ and $N' = \bigvee_\gamma \psi(\mathcal{J}_\gamma)$ and $N'' = \bigvee_\omega \psi(\mathcal{K}_\omega)$. Then $\psi(\mathcal{K}_\omega) \leq \psi(\mathcal{I}_\lambda)$ and $\psi(\mathcal{K}_\omega) \leq \psi(\mathcal{J}_\gamma)$ so $N'' = \bigvee_\omega \psi(\mathcal{K}_\omega) \leq \bigvee_\lambda \psi(\mathcal{I}_\lambda) = N$ and $N'' = \bigvee_\omega \psi(\mathcal{K}_\omega) \leq \bigvee_\gamma \psi(\mathcal{J}_\gamma) = N'$. Once we prove that N, N', and N'' are all maximal ideals we will know that $N = N' = N''$ so F_ψ is well defined.

First suppose that $N = \bigvee_\lambda \psi(\mathcal{I}_\lambda) \in NI(Y)$. This clearly cannot happen since then $\mathcal{I}_\lambda \leq \psi^{-1}(N) \in NI(X)$ for all $\lambda \in \Lambda$.

Thus suppose that $N = C^*(Y)$. This implies that $\bigcap_\lambda Z(\psi(\mathcal{I}_\lambda)) = \emptyset$. However since $\mathcal{I}_{\lambda_1} \leq \mathcal{I}_{\lambda_2}$ for $\lambda_1 \leq \lambda_2$ then $\psi(\mathcal{I}_{\lambda_1}) \leq \psi(\mathcal{I}_{\lambda_2})$ so $Z(\psi(\mathcal{I}_{\lambda_1})) \supset Z(\psi(\mathcal{I}_{\lambda_2}))$ is a directed collection of compact sets with the finite intersection property but with empty intersection. Thus there is an $\lambda_0 \in \lambda$ so that $\bigcap_{\lambda \leq \lambda_0} Z(\psi(\mathcal{I}_\lambda)) = \emptyset$ which implies that $Z(\psi(\mathcal{I}_\lambda)) = \emptyset$ and this contradicts $\psi(\mathcal{I}_\lambda) \in NI(Y)$. Thus $N \neq C^*(Y)$ so N is a maximal ideal in $C^*(Y)$. The proofs for N' and N'' are similar.

Now we show that F_ψ is a bijection. Consider $\psi^{-1} : NI(Y) \to NI(X)$ and construct $F_{\psi^{-1}}$ as above. We show that $F_{\psi^{-1}}$ is the inverse of F_ψ. Let $N = \bigvee_\lambda \psi(\mathcal{I}_\lambda) = F_\psi(M)$ for some $M \in MI(X)$. Then $\psi(\mathcal{I}_\lambda)$ is an increasing net in $NI(Y)$ with $\bigvee_\lambda \psi(\mathcal{I}_\lambda) = N$ so we can use $\psi(\mathcal{I}_\lambda)$ to compute $F_{\psi^{-1}}(N)$. Thus $M = \bigvee_\lambda \mathcal{I}_\lambda = \bigvee_\lambda \psi^{-1}(\psi(\mathcal{I}_\lambda)) = F_{\psi^{-1}}(N)$. Therefore $F_{\psi^{-1}}(F_\psi(M)) = M$. Similarly $F_\psi(F_{\psi^{-1}}(N)) = N$ so F_ψ and $F_{\psi^{-1}}$ are inverses and F_ψ is a bijection. ∎

Proposition 64 *The function F_ψ is a homeomorphism when we put the hull-kernel topology on $MI(X)$ and on $MI(Y)$.*

Proof: In Proposition 63 we have shown that F_ψ is a bijection. Thus all that remains is to show that F_ψ is continuous.

First we prove that if $I \in NI(X)$ and $M \in MI(X)$ with $Z(I)$ infinite and $I \subset M$ then $\psi(I) \subset F_\psi(M)$. Since $Z(I)$ is infinite, there is an increasing sequence $\{\mathcal{I}_n\}$ in $NI(X)$ with $\mathcal{I}_0 = I$ and $\bigvee_n \mathcal{I}_n = M$. Then clearly $\psi(I) \leq \bigvee_n \mathcal{I}_n = F_\psi(M)$.

Now we show that F_ψ is continuous. To do this we will show that $F_\psi(cl(A)) \subset cl(F_\psi(A))$. Now the closure of $A \subset MI(X)$ in the hull-kernel topology is the set

$$\{\mathcal{M} \in MI(X) | \mathcal{M} \supset \bigcap \{I | I \in A\}\}.$$

Thus

$$F_\psi(cl(A)) = \{F_\psi(M) | M \supset \bigcap \{I | I \in A\}\}$$

and

$$cl(F_\psi(A)) = \{N \in MI(Y) | N \supset \bigcap \{F_\psi(I) | I \in A\}\}.$$

If $A = \{M_1, M_2, M_3, \ldots, M_n\}$, then both A and $F_\psi(A)$ are closed so clearly $F_\psi(cl(A)) = F_\psi(A) = cl(F_\psi(A))$. Thus suppose that A is infinite. Then $\mathcal{I} = \bigcap\{I | I \in A\} \in NI(X)$ and $\mathcal{J} = \bigcap\{F_\psi(I) | I \in A\} \in NI(Y)$. Furthermore, $Z(\mathcal{I})$ is infinite.

Take $M \in MI(X)$ so that $M \supset \mathcal{I}$. We wish to show that $F_\psi(M) \supset \mathcal{J}$. Now $\mathcal{J} \subset F_\psi(I)$ for all $I \in A$ so $\psi^{-1}(\mathcal{J}) \subset F_{\psi^{-1}}(F_\psi(I)) = I$ for all $I \in A$. Thus $\psi^{-1}(\mathcal{J}) \subset \mathcal{I}$ so $\mathcal{J} \subset \psi(\mathcal{I})$. However, $\psi(\mathcal{I}) \subset F_\psi(M)$ so $\mathcal{J} \subset \psi(\mathcal{I}) \subset F_\psi(M)$. Thus if $M \in cl(A)$ then $F_\psi(M) \in cl(F_\psi(A))$ – so F_ψ is continuous.

It is well known that $MI(X)$ with the hull-kernel topology is a compact Hausdorff space ([GJ], 7M). This means that F_ψ is a homeomorphism, being a continuous bijection between two compact Hausdorff spaces. ∎

Theorem 65 *Let X and Y be compact Hausdorff spaces. Suppose that $CA(X)$ is lattice isomorphic to $CA(Y)$. Then $C^*(X)$ is isomorphic to $C^*(Y)$.*

Proof: From Propositions 62, 63, and 64 we know that $MI(X)$ is homeomorphic to $MI(Y)$. However, it is well know that $MI(X)$ with the hull-kernel topology is homeomorphic to X. Thus X and Y are homeomorphic. However, then clearly $C^*(X)$ and $C^*(Y)$ are isomorphic. ∎

We mention that the converse to this theorem is trivial.

Theorem 66 *Let X and Y be spaces. Then $K(X) \cong K(Y)$ if and only if $C^*(X)/C_0(X) \cong C^*(Y)/C_0(Y)$.*

Proof: One direction is easy, the other much harder. First, suppose $C^*(X)/C_0(X) \cong C^*(Y)/C_0(Y)$. Then by Lemma 60, we know $C^*(\beta X \setminus X) \cong C^*(\beta Y \setminus Y)$, so that $CA(\beta X \setminus X) \cong CA(\beta Y \setminus Y)$. However, then by Lemma 61, $PSA(X) \cong PSA(Y)$, but $PSA(X) \cong K(X)$ and similarly for Y. Thus, $K(X) \cong K(Y)$.

Conversely, suppose $K(X) \cong K(Y)$. Then, as before, $PSA(X) \cong PSA(Y)$ so, by Lemma 61, $CA(\beta X \setminus X) \cong CA(\beta Y \setminus Y)$. Now Theorem 65 then gives us that $C^*(\beta X \setminus X)$ is isomorphic to $C^*(\beta Y \setminus Y)$. However, then $C^*(X)/C_0(X)$ is isomorphic to $C^*(Y)/C_0(Y)$. ∎

The following diagram will help in illustrating the spaces and isomorphism involved.

$$\begin{tikzcd}
 & K(X) \arrow[r, "\cong"] \arrow[d, "\cong"'] & K(Y) \arrow[d, "\cong"] & \\
 & CA(\beta X \setminus X) \arrow[r, "\phi"] & CA(\beta Y \setminus Y) & \\
 & NI(\beta X \setminus X) \arrow[r, "\psi"] & NI(\beta Y \setminus Y) & \\
\beta X \setminus X \cong & MI(\beta X \setminus X) \arrow[r, "F_\psi"] & MI(\beta Y \setminus Y) & \cong \beta Y \setminus Y
\end{tikzcd}$$

Figure 8: Diagram for Theorem 66

Remark 1 *It is well known that if S and T are compact Hausdorff spaces, then S is homeomorphic to T if and only if $C^*(S) \cong C^*(T)$ [GJ]. This fact and Lemma 60 imply that Magill's Theorem and Theorem 66 are equivalent.*

Let us compare the preceeding proof with Magill's. If we assume that $\beta X \setminus X$ is homeomorphic to $\beta Y \setminus Y$, then it is easy to show that $K(X)$ is isomorphic to $K(Y)$. The other direction is, again, the hard one. So we assume that $K(X)$ is isomorphic to $K(Y)$ and show that $\beta X \setminus X$ is homeomorphic to $\beta Y \setminus Y$. In [MG2], Magill first defines a "dual point" of a lattice and proves some results about these. Using the fact that a dual point in $K(X)$ has to go to a dual point in $K(Y)$ under a lattice isomorphism, he gets a mapping $f : \beta X \setminus X \to \beta Y \setminus Y$. He then proves that f is a bijection by constructing an inverse (using the same methods as he for f). Next he proves that f is continuous by showing that it is

a closed map. His construction requires quite extensive investigation into the structure of $K(X)$.

What we have done essentially boils down to the following. Any ideal $\mathcal{I}$ in $C^*(H)$ (for compact Hausdorff H) is of the form

$$\mathcal{I} = \{f \in C^*(H) |\ f|_S = 0\}$$

for some closed $S \subset H$. Using this fact, we can translate statements about ideals in $C^*(H)$ into statements about closed sets in H. We used the isomorphism between $K(X)$ and $K(Y)$ to get information about how closed sets (ideals) in $\beta X \setminus X$ get mapped to closed sets (ideals) in $\beta Y \setminus Y$. Since $\beta X \setminus X$ is compact and Hausdorff, any point $x \in \beta X \setminus X$ is the intersection of all the closed sets that contain it (this is the translation of the fact that a maximal ideal is uniquely determined by the collection of all non-maximal ideals which it contains). So we use the mapping between closed sets to get a mapping F_Ψ between points. The fact that we have a lattice mapping makes this work. Now, clearly F_Ψ has to be continuous since it preserves the closed sets which contain a point, and the closed sets in question are a base for the closed sets (this corresponds to the mapping F_Ψ being continuous in the hull-kernel topology because the hull-kernel topology is defined by order and so was F_Ψ).

Using the algebra, we were able to do all of the manipulations without explicitly looking at closed sets or points – the algebra did most of the work.

As mentioned in the proof of Theorem 66, proving that $C^*(X)/C_0(X) \cong C^*(Y)/C_0(Y)$ implies that $K(X) \cong K(Y)$ is easy. It is the converse that is difficult. Another way of stating this converse is that the lattice of all closed unital proper subalgebras of $C^*(H)$ (for compact Hausdorff H) determines $C^*(H)$. In our case, $H = \beta X \setminus X$.

Why is it necessary to look at only the unital algebras? What about the lattice of all closed subalgebras? In order to understand the answer to this, we need to know something about what a general closed subalgebra of $C^*(X)$ looks like.

Theorem 67 *Let $\mathcal{A}$ be a closed subalgebra of $C^*(X)$ with X a compact Hausdorff space.*

Then there is a compact Hausdorff space T and a quotient map $\pi : X \to T$ so that $\mathcal{A}$ is one of the following:

1. *$\mathcal{A}$ is an ideal.*

   or

2. *If $\mathcal{A}$ contains the constants, then $\mathcal{A} = \pi^*(C^*(T))$*

   or

3. *$\mathcal{A} = \pi^*(\mathcal{I})$ for some closed ideal $\mathcal{I} \subset C^*(T)$.*

Proof: Notice that type 1 and type 3 are really the same (if we let $T = X$).

First, we need to find the space T. We define T as the collection of all the maximal stationary sets of $\mathcal{A}$, so that T is a partition of X. We topologize T using the quotient topology by the natural quotient map $\pi : X \to T$. Defined this way, T is compact and Hausdorff.

Clearly all of these are subalgebras. Thus suppose that $\mathcal{A}$ is not of type 1 or 2. What we need to show is that it is of type 3.

In this case, $1 \notin \mathcal{A}$. Let $\mathcal{C} = \mathcal{A} \oplus 1 \subset C^*(X)$. First we need to show that $\mathcal{C}$ is closed, so that it is a C^*-algebra and thus of the form $C^*(T)$ for some compact Hausdorff T.

There is only one C^*-algebra norm on any unital *-algebra, so we have a unique choice of norm for $\mathcal{C} = \mathcal{A} \oplus 1$. In fact, this norm will extend the norm on $\mathcal{A}$. Define the homomorphism $\phi : \mathcal{C} \to \mathbb{R}$ by $\phi(f + c) = c$ where $f \in \mathcal{C}$ and $c \in \mathbb{R}$. Then clearly $\mathcal{A} \subset \mathcal{C}$ is the kernel of ϕ so $\mathcal{A}$ is a closed ideal in $\mathcal{C}$ (in fact, $\mathcal{A}$ is a maximal ideal in $\mathcal{C}$). Thus, $\mathcal{C}$ must be closed.

Now that we know that $\mathcal{C}$ is closed, we know that $\mathcal{C} \cong C^*(T)$ (in fact by the Stone-Weierstrass Theorem, $\mathcal{C}$ is exactly the set of functions which are constant on elements of T). Thus, there is some $p \in T$ so that $\mathcal{A} = \{f \in C^*(X) | f|_p = 0\}$. This is the same thing as $\mathcal{A} = \pi^*(\mathcal{I})$ with $\mathcal{I} = \{f \in C^*(T) | f(p) = 0\}$. ∎

Let $\mathcal{C}$ be a closed unital subalgebra. Then there are many non-unital closed subalgebras $\mathcal{A}$ so that $\mathcal{C} = \mathcal{A} \oplus 1$ (in fact any such $\mathcal{A}$ will be a maximal ideal in $\mathcal{C}$). Now, for each compactification αX, there corresponds a unique closed unital subalgebra C_α. However, then there will be many closed non-unital subalgebras corresponding to αX. Thus, the lattice of all closed subalgebras of $C^*(X)$ is much larger than the lattice of all compactifications of X. This is why we need to assume that the algebras are unital in Theorem 66.

We can generalize Theorem 66 some to the situation where we have a sub-lattice of $K(X)$ lattice isomorphic to a sub-lattice of $K(Y)$. Here, the two sub-lattices must be intervals, and the lower bound of each interval must satisfy some additional hypothesis.

We set up some notation for the statement of the following theorem.

Notation 14 *If $\alpha X \leq \delta X \in K(X)$, then $K(X, \alpha, \delta) = \{\nu X \in K(X) | \alpha X \leq \nu X \leq \delta X\}$*

Remark 2 *Note that $K(X, \alpha, \delta)$ is also a complete lattice, being an interval in the complete lattice $K(X)$.*

We recall a definition from Chapter 2.

Definition 15 *For $\alpha X \in K(X)$, we define $C_\alpha(X) = \{f \in C^*(X) | f$ extends to $\alpha X\}$.*

Now we record a fact which we need in the proofs of several of the succeeding results.

Lemma 68 *Let X be a (not necessarily locally compact) space and $\alpha X \leq \delta X \in K(X)$. Then $K(X, \alpha, \delta)$ is lattice isomorphic to the collection of all closed unital subalgebras of C_δ which contain C_α. If $C_\alpha = \mathcal{I} \oplus 1$ for some ideal $\mathcal{I}$, then this is isomorphic to the lattice of all closed unital algebras of $C_\delta/\mathcal{I}$.*

Proof: The isomorphism between $K(X)$ and $PSA(X)$ carries the interval $K(X, \alpha, \delta)$ bijectively onto the set of all closed algebras between C_α and C_δ.

Once we note this, the proof of the second remark is similar to the proof of Lemma 61.

We never explicitly used the fact that X was locally compact in the proofs of Lemmas 60 and 61. The only place it was used is in the fact that $C_0(X)$ separates points from closed

sets. We simply replace $C_0(X)$ by $\mathcal{I}$ and the same proofs go through when X is not locally compact. ∎

Theorem 69 *Let X and Y be spaces $\alpha X \leq \delta X \in K(X)$ and $\gamma Y \leq \theta Y \in K(Y)$. Suppose that αX and γY are such that $C_\alpha = \mathcal{I} \oplus 1$ and $C_\gamma = \mathcal{J} \oplus 1$ for some ideals $\mathcal{I}$ and $\mathcal{J}$. Then $K(X, \alpha, \delta) \cong K(Y, \gamma, \theta)$ if and only if $C_\delta(X)/\mathcal{I} \cong C_\theta(Y)/\mathcal{J}$.*

Proof: The proof is similar to the proof of Theorem 66. By Lemma 68 we know that $K(X, \alpha, \delta)$ is isomorphic to the set of all closed unital subalgebras of C_δ which contain $\mathcal{I}$ which in turn is isomorphic to the set of all closed unital subalgebras in $C_\delta/\mathcal{I}$. Also note that Lemma 60 should be replaced with $C_\delta/\mathcal{I} \cong C^*(T)$ where $T \subset \delta X$ is the unique non-singleton set in $(\pi_\alpha^\delta)^{-1}(\alpha X)$, and π_α^δ is the canonical projection of δX onto αX which preserves X. Once we make these changes, all that remains is to replace $C^*(X)$ everywhere in the proof with C_δ, $C^*(Y)$ with C_θ, $C_0(X)$ with $\mathcal{I}$, and $C_0(Y)$ with $\mathcal{J}$. ∎

If ωX is the one-point compactification of X, then $C_\omega = C_0(X) \oplus 1$. Thus, the one point compactification satisfies the additional hypothesis that is needed in the generalization of Theorem 66. Further, clearly $\omega X \leq \beta X$. Thus Theorem 66 is a special case of Theorem 69.

We can get another slight generalization to the case where the lower bounds of the intervals are not of the form $C_\alpha = \mathcal{I} \oplus 1$. However, we do need some hypotheses on the lower bounds.

Theorem 70 *Let $\alpha_i X \in K(X)$ for $i = 1, \ldots, n$ with $C_{\alpha_1} = \mathcal{I}_i \oplus 1$ for some ideals $\mathcal{I}_i \subset C_\delta$ with $\mathcal{I}_i \bigvee \mathcal{I}_j = C_\delta$ if $i \neq j$. Let $\mathcal{I} = \bigcap I_i$ and $\alpha X \leq \delta X \in K(X)$ with $\alpha X = \bigwedge \alpha_i X$.*

If $K(X, \alpha, \delta) \cong K(Y, \gamma, \theta)$ for some $\gamma Y \leq \theta Y \in K(Y)$, then there are $\gamma_i Y \in K(Y)$ so that $\gamma Y = \bigwedge \gamma_i Y$ with $C_{\gamma_i} = \mathcal{J}_i \oplus 1$ for some ideals $\mathcal{J}_i \subset C_\theta$. Furthermore, letting $\mathcal{J} = \bigcap \mathcal{J}_i$, we have $C_\delta/\mathcal{I} \cong C_\theta/\mathcal{J}$.

Conversely, if there is some ideal $\mathcal{J}$ with $C_0(Y) \subset \mathcal{J}$ and such that $C_\delta/\mathcal{I} \cong C_\theta/\mathcal{J}$, then $K(X, \alpha, \delta) \cong K(Y, \gamma, \theta)$ where $\gamma Y = \bigwedge \gamma_i Y$ with $C_{\gamma_i} = \mathcal{J}_i \oplus 1$ for some ideals $\mathcal{J}_i \subset C_\theta$.

Furthermore, $\mathcal{J}_i \bigvee \mathcal{J}_j = C_\theta$ *if* $i \neq j$*, and* $\mathcal{J} = \bigcap \mathcal{J}_i$*.*

Proof: First, suppose that $K(X, \alpha, \delta) \cong K(Y, \gamma, \theta)$ and let us denote this isomorphism by Γ. Then clearly there are $\gamma_i Y \in K(Y, \gamma, \theta)$ so that $\gamma Y = \bigwedge \gamma_i Y$ and $\Gamma(\alpha_i X) = \gamma_i Y$ (since Γ is an isomorphism). Furthermore by Proposition 62 we know that $C_{\gamma_i} = \mathcal{J}_i \oplus 1$ for some ideals $\mathcal{J}_i \subset C_\theta$ since $C_{\alpha_i} = \mathcal{I}_i \oplus 1$. Thus, the only thing left to show is that $C_\delta/\mathcal{I} \cong C_\theta/\mathcal{J}$. Since $\alpha X \leq \alpha_i X \leq \delta X$, we know that $K(X, \alpha_i, \delta)$ is a sublattice of $K(X, \alpha, \delta)$ so that $K(X, \alpha_i, \delta) \cong K(Y, \gamma_i, \theta)$ under Γ. Therefore, by Theorem 69, $C_\delta/\mathcal{I}_i \cong C_\theta/\mathcal{J}_i$ for $i = 1, \ldots, n$. However, then by Lemma 59 (Chinese Remainder Theorem), $C_\delta/\mathcal{I} \cong C_\delta/\mathcal{I}_1 \oplus \cdots \oplus C_\delta/\mathcal{I}_n$ and similarly for $C_\theta/\mathcal{J}$ (this is since $\mathcal{I}_i \bigvee \mathcal{I}_j = C_\delta$ if $i \neq j$). Thus, $C_\delta/\mathcal{I} \cong C_\theta/\mathcal{J}$.

Conversely, suppose that $C_\delta/\mathcal{I} \cong C_\theta/\mathcal{J}$. Let $\alpha' X \in K(X)$ be such that $C_{\alpha'} = \mathcal{I} \oplus 1$; and similarly, let $\gamma' Y \in K(Y)$ so that $C_{\gamma'} = \mathcal{J} \oplus 1$ (this is possible since $\mathcal{I}$ separates points from closed sets in X and $\mathcal{J}$ separates points from closed sets in Y). Then, by Theorem 69, $K(X, \alpha', \delta) \cong K(Y, \gamma', \theta)$. However, $\alpha' X \leq \alpha X$ since $\mathcal{I} \subset \mathcal{I}_i$ and, similarly, $\gamma' Y \leq \gamma Y$. Thus, $K(X, \alpha, \delta)$ is a sublattice of $K(X, \alpha', \gamma)$ and $K(Y, \gamma, \theta)$ is a sublattice of $K(Y, \gamma', \theta)$ and the isomorphism between $K(X, \alpha', \delta)$ and $K(Y, \gamma', \theta)$ takes $K(X, \alpha, \delta)$ to $K(Y, \gamma, \theta)$. Thus, $K(X, \alpha, \delta) \cong K(Y, \gamma, \theta)$. ∎

Very roughly, Theorem 70 says that if you have an isomorphism between $K(X, \alpha, \delta)$ and $K(Y, \gamma, \theta)$ where αX is the meet of finitely many compactifications with disjoint "stationary sets", then you can extend this isomorphism all the way down to the compactification $\alpha' X$, which is the compactification with the "stationary set" consisting of the union of the "stationary sets" of the $\alpha_i X$'s.

We could extend this theorem to infinitely many $\alpha_i X$'s under the assumption that if $C_\delta/\mathcal{I}_i \cong C_\theta/\mathcal{J}_i$ for all i in some indexing set Λ, then $C_\delta/\mathcal{I} \cong C_\theta/\mathcal{J}$ where $\mathcal{I} = \bigcap \mathcal{I}_i$ and $\mathcal{J} = \bigcap \mathcal{J}_i$. However, this assumption is not true in general if there are infinitely many $\mathcal{I}_i$'s. Here is an example of this.

Example Let $Y = X = I\!R$ and $\delta X = \theta Y$ be such that $\delta X \setminus X = [0,1]$ (we know that such compactifications exist, e.g. see [AB]). Let $S_i = \{1/i\} \subset [0,1]$ for $i = 1,2,3,4,\ldots$ and $S_0 = \{0\}$. Let $T_{2i} = \{1/i\}$ and $T_{2i+1} = \{1 - 1/i\}$ for $i = 1,2,3,4,\ldots$, and $T_0 = \{0\}$ and $T_1 = \{1\}$. Now, let $\mathcal{I}_i = \{f \in C_\delta |\ f|_{S_i} = 0\}$ and $\mathcal{J}_i = \{f \in C_\theta |\ f|_{T_i} = 0\}$. Finally, let $\mathcal{I} = \bigcap \mathcal{I}_i$ and $\mathcal{J} = \bigcap \mathcal{J}_i$. Then clearly $C_\delta/\mathcal{I}_i \cong C_\theta/\mathcal{J}_i$ for all i. However, $C_\delta/\mathcal{I}$ is not isomorphic to $C_\theta/\mathcal{J}$ since $cl(\bigcup S_i)$ is not homeomorphic to $cl(\bigcup T_i)$. Notice that the conditions $\mathcal{I}_i \bigvee \mathcal{I}_j = C_\delta$ and $\mathcal{J}_i \bigvee \mathcal{J}_i = C_\theta$ are trivially satisfied.

The only way that the Chinese Remainder Theorem will work for a collection of ideals $\{\mathcal{I}_i\}$ indexed by some set Λ, is for the S_i's to be "discrete" (*i.e.* not to have any limit points). However, this is not possible in a compact space. Thus, one would **not** expect the Chinese Remainder Theorem to extend to infinitely many ideals.

The proof of Theorem 70 uses the Chinese Remainder Theorem to show that $C_\delta/\mathcal{I} \cong C_\theta/\mathcal{J}$. However, this is not the only way to show this. What you really need are the following conditions on αX and γX:

1. $\alpha X = \bigwedge \alpha_i X$ for some collection $\alpha_i X \in K(X)$ with $C_{\alpha_i} = \mathcal{I}_i \oplus 1$ and $\mathcal{I}_i \subset C_\delta$ ideals (with $i \in \Lambda$, some indexing set).

2. $\mathcal{I}_i \vee \mathcal{I}_j = C_\delta$ if $i \neq j$.

3. $\gamma Y = \bigwedge \gamma_i Y$ for some collection $\gamma_i Y \in K(Y)$ with $C_{\gamma_i} = \mathcal{J}_i \oplus 1$ with $\mathcal{J}_i \subset C_\theta$ ideals (with $i \in \Lambda$, same indexing set).

4. $\mathcal{J}_i \vee \mathcal{J}_j = C_\theta$ if $i \neq j$.

5. $C_\delta/\mathcal{I} \cong C_\theta/\mathcal{J}$ if and only if $C_\delta/\mathcal{I}_i \cong C_\theta/\mathcal{J}_i$ for all $i \in \Lambda$.

Notice that the indexing set has to be the same for αX and γY. We got this for free in the finite case.

Condition 5 is very strong. If Λ is finite, this condition is guaranteed by the Chinese Remainder Theorem. However, for infinite Λ this puts conditions on both αX and on γY.

This conditions is a "compatibility" condition on the indexing of the $\alpha_i X$'s and the $\gamma_i Y$'s. Furthermore, these conditions are not generically true, as we have seen from the previous example.

4.4 Non Locally Compact Spaces

If X is not locally compact, $K(X)$ is not a complete lattice. This is because X has no one-point compactification. In our algebraic setting, this translates into there being no minimal ideal in $C^*(X)$ which separates points from closed sets. Thus, there is no ideal to replace $C_0(X)$ in Theorem 66. However, we can get some results like Theorems 69 and 70.

Notation 16 *R(X) is the set of all points in X which have no compact neighborhood.*

Note that $R(X) = cl(\alpha X \setminus X) \cap X$ for any compactification αX. As we mentioned, this is not the same as the $R(X)$ we used in Chapter 3.

The main trouble in extending Theorem 66 to the general case (not necessarily locally compact), is that $C_0(X)$ fails to distinguish points of $R(X)$ from points of $\alpha X \setminus X$ for any compactification αX. If we "stay away" from $R(X)$, we have no trouble. This is the case if we start with a compactification αX and look at all the compactifications larger than it. However, just like Theorems 69 and 70, we need some conditions on αX (there is no reason something should be true in general if it is not true for a special case).

Theorem 71 *Let X and Y be (not necessarily locally compact) spaces, $\alpha X \leq \delta X \in K(X)$ and $\gamma Y \leq \theta Y \in K(Y)$. Suppose that αX and γY are such that $C_\alpha = \mathcal{I} \oplus 1$ and $C_\gamma = \mathcal{J} \oplus 1$ for some ideals $\mathcal{I} \subset C_\delta$ and $\mathcal{J} \subset C_\theta$. Then $K(X, \alpha, \delta) \cong K(Y, \gamma, \theta)$ if and only if $C_\delta(X)/\mathcal{I} \cong C_\theta(Y)/\mathcal{J}$.*

Proof: By Lemma 68, $K(X, \alpha, \delta)$ is order isomorphic to the collection of all closed unital subalgebras of $C_\delta/\mathcal{I}$. Similarly, $K(Y, \gamma, \theta)$ is order isomorphic to the collection of all closed unital subalgebras of $C_\theta/\mathcal{J}$.

First, suppose that $K(X, \alpha, \delta) \cong K(Y, \gamma, \theta)$. Then the lattice of all closed unital subalgebras of $C_\delta/\mathcal{I}$ is isomorphic to the lattice of all closed unital subalgebras of $C_\theta/\mathcal{J}$. Now we

are in exactly the same situation as in the proof of Theorem 66 for the converse (the hard direction). The only difference is that we know that $CA(S) \cong CA(T)$, where S is the only non-singleton set in the partition $(\pi_\alpha^\delta)^{-1}$ of δX and T is the only non-singleton set in the partition $(\pi_\gamma^\theta)^{-1}$ of θY. (In the proof of Theorem 66, we knew that $CA(\beta X \backslash X) \cong CA(\beta Y \backslash Y)$).

Conversely, suppose that $C_\delta/\mathcal{I} \cong C_\theta/\mathcal{J}$. Clearly, then the lattice of all closed unital subalgebras of $C_\delta/\mathcal{I}$ is isomorphic to the lattice of all closed unital subalgebras of $C_\theta/\mathcal{J}$. Thus, $K(X, \alpha, \delta) \cong K(Y, \gamma, \theta)$. ∎

We also get a version of Theorem 70 in the general case, where X is not necessarily locally compact.

Theorem 72 *Let X and Y be (not necessarily locally compact) spaces.*

Let $\alpha_i X \in K(X)$ for $i = 1, \ldots, n$ with $C_{\alpha_i} = \mathcal{I}_i \oplus 1$ for some ideals $\mathcal{I}_i \subset C_\delta$ with $\mathcal{I}_i \bigvee \mathcal{I}_j = C_\delta$ if $i \neq j$. Let $\mathcal{I} = \bigcap I_i$ and $\alpha X \leq \delta X \in K(X)$ with $\alpha X = \bigwedge \alpha_i X$.

If $K(X, \alpha, \delta) \cong K(Y, \gamma, \theta)$ for some $\gamma Y \leq \theta Y \in K(Y)$, then there are $\gamma_i Y \in K(Y)$ so that $\gamma Y = \bigwedge \gamma_i Y$ with $C_{\gamma_i} = \mathcal{J}_i \oplus 1$ for some ideals $\mathcal{J}_i \subset C_\theta$. Furthermore, letting $\mathcal{J} = \bigcap \mathcal{J}_i$, we have $C_\delta/\mathcal{I} \cong C_\theta/\mathcal{J}$.

Conversely, if there are compactifications $\gamma' Y \leq \theta Y \in K(Y)$ with $C_{\gamma'} = \mathcal{J} \oplus 1$ for some ideal $\mathcal{J} \subset C_\theta$ and such that $C_\theta/\mathcal{J} \cong C_\delta/\mathcal{I}$ then $K(X, \alpha, \delta) \cong K(Y, \gamma, \theta)$ for some $\gamma Y \geq \gamma' Y$. Furthermore, there are compactifications $\gamma_i Y \in K(Y)$ so that $C_{\gamma_i} = \mathcal{J}_i \oplus 1$ for some ideals $\mathcal{J}_i \subset C_\theta$ and $\mathcal{J} = \bigcap \mathcal{J}_i$ and $\mathcal{J}_i \bigvee \mathcal{J}_j = C_\theta$ for $i \neq j$ and $\gamma Y = \bigwedge \gamma_i Y$.

Proof: First suppose that $K(X, \alpha, \delta) \cong K(Y, \gamma, \theta)$. Then since $\alpha X \leq \alpha_i X \leq \delta X$ clearly there are compactifications $\gamma_i Y$ with $\gamma Y \leq \gamma_i Y \leq \theta Y$ so that $K(X, \alpha_i, \delta) \cong K(Y, \gamma_i, \theta)$. Further by Proposition 62 we see that $C_{\gamma_i} = \mathcal{J}_i \oplus 1$ for some closed ideals $\mathcal{J}_i \subset C_\theta$. By Theorem 71 we have $C_\delta/\mathcal{I}_i \cong C_\theta/\mathcal{J}_i$ for $i = 1, \ldots, n$. Then by Lemma 59 (the Chinese Remainder Theorem) $C_\delta/\mathcal{I} \cong C_\delta/\mathcal{I}_1 \oplus \cdots C_\delta/\mathcal{I}_n \cong C_\theta/\mathcal{J}_1 \oplus \cdots \oplus C_\theta/\mathcal{J}_n \cong C_\delta/\mathcal{J}$ where $\mathcal{J} = \bigcap \mathcal{J}_i$.

Conversely, suppose that $C_\delta/\mathcal{I} \cong C_\theta/\mathcal{J}$. Let $\alpha' X \in K(X)$ be so that $C_{\alpha'} = \mathcal{I} \oplus 1$. Then by Theorem 71 we know that $K(X, \alpha', \delta) \cong K(Y, \gamma', \theta)$. However, $\alpha' X \leq \alpha X \leq \delta X$ and

$\gamma' Y \leq \gamma Y \leq \theta Y$ so $K(X, \alpha, \delta)$ is a sublattice of $K(X, \alpha', \delta)$ and similarly $K(Y, \gamma, \theta)$ is a sublattice of $K(Y, \gamma', \theta)$. Thus, $K(X, \alpha, \delta) \cong K(Y, \gamma, \theta)$. The statement about the existence of $\gamma_i Y \in K(Y)$ follows from the isomorphism between $K(X, \alpha', \delta)$ and $K(Y, \gamma', \theta)$. ∎

One might think that you could "approximate" $K(X)$ and $K(Y)$ by sequences $K(X, \alpha_n, \beta)$ and $K(Y, \gamma_n, \beta)$ where $\{\alpha_n X\}$ and $\{\gamma_n Y\}$ are decreasing sequences of compactifications in $K(X)$ and $K(Y)$ respectively. Then you could try to use Theorems 71 and 72 to get an isomorphism between $K(X)$ and $K(Y)$. However, this is not possible in general as the next example clearly shows.

Example Let $X = (0, 1] \cup ([2, 3] \cap \mathbf{Q})$, where $\mathbf{Q}$ is the set of rational numbers. Let $Y = (0, 1]$. Let αX and γY be compactifications of X and Y respectively, with $C_\alpha = \mathcal{I} \oplus 1$ and $C_\gamma = \mathcal{J} \oplus 1$ for some ideals $\mathcal{I} \subset C^*(X)$ and $\mathcal{J} \subset C^*(Y)$. Then $Z(\mathcal{I}) \subset cl_{\beta X}((0, 1])$ (since $\mathcal{I}$ must separate points in X). Thus from $C^*(X)/\mathcal{I}$, we only get information about compactifications of the (0,1] part of X, since $C^*(X)/\mathcal{I} \cong C^*(Z(\mathcal{I}))$ by Proposition 58. So, if we have decreasing sequences $\alpha_n X$ and $\gamma_n Y$ so that $K(X, \alpha_n, \beta) \cong K(Y, \gamma_n, \beta)$, this does **not** say that $K(X) \cong K(Y)$.

What we can say is the following. Since $X = (0, 1] \cup ([2, 3] \cap \mathbf{Q})$, we know that $C^*(X) = C^*((0, 1]) \oplus C^*([2, 3] \cap \mathbf{Q})$. Let $\delta X \in K(X)$ be such that $C_\delta = (C_0((0, 1]) \oplus 1) \oplus C^*([2, 3] \cap \mathbf{Q})$. Then we can say that $K(X, \delta, \beta) \cong K(Y)$. Roughly speaking, this just says that we can only approximate that part of $K(X)$ which "lies above" the (0,1] part of X. This is not surprising, however. Somehow we should have to take into account the fact that X is not locally compact, since $K(X)$ certainly does.

Notice that the method of proof for Theorem 71 and Theorem 72 is very similar to the proofs of Theorems 69 and 70 respectively. The main difference between the theorems for the locally compact case and the non locally compact case is in the hypotheses. We need slightly more hypotheses in the non locally compact case in order to guarantee that the ideals correspond to compactifications. For the locally compact case, any ideal $\mathcal{I}$ which contains $C_0(X)$ will correspond to a compactification (since $\mathcal{I} \oplus 1$ will be a closed unital

subalgebra which separates points from closed sets). For X a non locally compact space, the situation is a little more complicated. For example, let $X = [0,1] \times [0,1] \setminus \{1\} \times [0,1)$, then $y = \{1\} \times \{1\}$ does not have any compact neighborhoods in X. Let $S = cl(\beta X \setminus X)$ and $\mathcal{I} = \{f | f|_S = 0\}$. Then $\mathcal{I}$ is an ideal, but $\mathcal{I} \oplus 1$ does not separate y from the closed set $[0, 1/2] \times [1/2, 1)$. Thus, $\mathcal{I} \oplus 1$ does not correspond to a compactification.

4.5 Categorical Perspective

Now we will step back a little and look at the global picture. For this section, we will presume some familiarity with the language of Category Theory (mainly the notion of a category and a functor). One possible reference is [MAC]. Let $CompHaus$ be the category of compact Hausdorff spaces with continuous functions as the morphisms, C^*Alg be the category of Abelian C^*-algebras with continuous algebra homomorphisms as the morphisms, and PO be the category of partially ordered sets with order-preserving functions as the morphisms. Then in this language, we have a contravariant functor $C^* : CompHaus \to C^*Alg$ which takes the space X to the C^*-algebra $C^*(X)$. Since X is homeomorphic to Y if and only if $C^*(X)$ is isomorphic to $C^*(Y)$, we know that the functor C^* is an isomorphism between $CompHaus$ and C^*Alg.

Let X be a compact Hausdorff space. Consider $P(X) =$ *lattice of all quotient spaces of* X where $S \leq T$ in $P(X)$ if there is some quotient map from T onto S. This should be a familiar lattice, since it is, in essence, the same thing we did for $K(X)$. In fact $P(\beta X \setminus X) = K(X)$ for a locally compact X. Now, if we know all the spaces in $P(X)$, then we can take the inverse limit ($P(X)$ is clearly a directed set and the bonding maps are compatible) of the inverse system $P(X)$ and we will get the space X. Applying the functor C^* to this system we get $CA(X)$ – the lattice of all closed unital subalgebras of $C^*(X)$ –and the projections in $P(X)$ become injections. Again, if we know all of the subalgebras in $CA(X)$ (as well as the lattice structure) and take a direct limit we will get $C^*(X)$.

Is there some way to get X as a direct limit? Yes, we just take $I(X)$ to be the lattice of all closed subspaces of X where $S \leq T$ if $S \subset T$. Using $I(X)$ as the directed set we get that

X is the direct limit of $I(X)$. Now, when we apply the functor C^* to $I(X)$ we get $PA(X)$ = *lattice of all projections of* $C^*(X)$ – *i.e.* the lattice of all $(\pi, C^*(S))$ where S is compact and Hausdorff and $\pi : C^*(X) \to C^*(S)$ is a surjective algebra morphism. Here we say that $p_1 \leq p_2$ (with $p_1 : C^*(X) \to \mathcal{A}$ and $p_2 : C^*(X) \to \mathcal{B}$ projections) if there is some $q : \mathcal{B} \to \mathcal{A}$ so that $p_1 = q \circ p_2$ which happens if and only if $ker(p_2) \subset ker(p_1)$. This is a natural way to define the order – it is similar to the way we defined the order in $P(X)$.

Now if all we know is the lattice structure of $I(X)$ (and not the spaces in $I(X)$) then it is easy to recover X since if $x \in X$ then $\{x\} \in I(X)$ is an atom (or, if we have $\emptyset \in I(X)$, then we just take all the elements in $I(X)$ which are immediately bigger than $\emptyset$). We can also recover the topology of X using the fact that $cl(A) = \bigwedge\{B \in I(X) | A \leq B\}$.

If we now use the functor C^* on $I(X)$, we can easily recover $C^*(X)$ from the lattice structure of $PA(X)$. Each maximal ideal in $C^*(X)$ corresponds to a dual points in $PA(X)$, so we can get the maximal ideals. In fact, we know the lattice structure of the closed ideals since a closed ideal is the kernel of an algebra homomorphism and we know the lattice structure of the algebra homomorphisms. Thus, we know the maximal ideals space and consequently we know $C^*(X)$.

On the other hand, for $P(X)$, it is more difficult to get the points of X since we are taking projections rather than injections. What Magill's Theorem (Theorem 57) says is that it is sufficient to know just the lattice structure of $P(X)$ in order to determine X. This is the same thing as saying that the functor $LP : CompHaus \to PO$ which takes the compact Hausdorff space X to $P(X)$ is a faithful functor (injective).

Similarly when we apply the functor C^* to $P(X)$ we get $CA(X)$ and what our Theorem 65 states is that the lattice structure of $CA(X)$ is sufficient to determine $C^*(X)$. As in the case of $P(X)$, it is not so easy to get information about the closed ideals in $C^*(X)$.

So, what we have is that the functors defined by $X \mapsto P(X)$ and $X \mapsto I(X)$ are faithful. Then simply composing with the functor C^* we get that the functors defined by $X \mapsto CA(X)$ and $X \mapsto PA(X)$ are also faithful. Furthermore, we know that the functors defined by $C^*(X) \mapsto CA(X)$ and $C^*(X) \mapsto PA(X)$ are faithful as well.

Now we will trace through the details of all the above equivalences and the direct limit and inverse limit constructions.

Inverse limit construction of X

First we do the inverse limit construction. Let $P(X)$ be as before and $\hat{P}(X) = P(X) \setminus \{X\}$. Furthermore, for $\mathcal{A} \leq \mathcal{B} \in P(X)$, let π_A^B be the canonical projection from B onto A. We claim that $\varprojlim P(X) = X$.

Notice that $\hat{P}(X)$ is not a directed set, so we cannot compute $\varprojlim \hat{P}(X)$. This is easy to see since if we let $x \neq y \neq z \in X$ and consider the two elements of $\hat{P}(X)$ given by $A = X/x \sim y$ and $B = X/x \sim z$. The clearly $\max\{A, B\} = X \notin \hat{P}(X)$.

From the definition of the inverse limit, $\varprojlim P(X) \subset \prod\{A | A \in P(X)\}$ and is the subset which satisfies the condition $(\pi_A^B(z))_A = (z)_A$ (where $(z)_A$ is the Ath coordinate of z).

To show that $X = \varprojlim P(X)$ we define the map

$$\Pi : X \to \prod\{A | A \in P(X)\}$$

by

$$(\Pi(x))_A = \pi_A^X(x).$$

Clearly Π is a continuous surjection onto every coordinate space. What we show is that $\Pi(X)$ is dense in $\varprojlim P(X)$, which would imply that Π is surjective (since X is compact). First we show that $\Pi(X) \subset \varprojlim P(X)$. To see this just notice that

$$(\Pi(x))_A = \pi_A^X(x) = \pi_A^B(\pi_B^X(x)) = \pi_A^B((\Pi(x))_B)$$

Let $z \in \varprojlim P(X)$. Let $\{A_i\} \subset P(X)$ be a finite collection and let $U_i \subset A_i$ be open neighborhoods of $(z)_{A_i}$ in A_i. Let $B \in P(X)$ so that $B \geq A_i$ for all i. Then there is some point $y \in B$ so that $(\pi_{A_i}^B(y))_{A_i} = (z)_{A_i}$ for each i. Choose any point $x \in (\pi_B^X)^{-1}(y)$ then $(\Pi(x))_{A_i} \in U_i$ for all i so $\Pi(X)$ is dense in $\varprojlim P(X)$. We claim that Π is also injective. Suppose not, then there are $x, y \in X$ so that $\Pi(x) = \Pi(y)$ or $(\pi_A^X(x))_A = (\pi_A^X(y))_A$ for each $A \in P(X)$. However, there is clearly a quotient map $q : X \to B$ which separates x from y

so this is impossible. Thus Π is injective. Since Π is both surjective and injective, it is a homeomorphism (X is a compact space and $\varprojlim P(X)$ is Hausdorff).

When we apply the functor C^* to $\hat{P}(X)$ we get $CA(X) \setminus \{C^*(X)\}$ and this is also not a directed set (since $\hat{P}(X)$ is not a directed set). This is what causes many of the complications in the proof of Theorem 65.

However, if all we know is the order structure of $P(X)$ we cannot use the inverse limit construction to recapture X (since the inverse limit requires knowledge of the points of each $A \in P(X)$).

Direct limit construction of $C^*(X)$

Now we will "dualize" the above inverse limit construction and show how $C^*(X)$ is the direct limit of $CA(X)$. One way to do this is simply to apply the covariant functor C^* to our above construction. However, this is not very illuminating. We will explicitly show that $C^*(X) = \varinjlim CA(X)$ and leave it to the reader to see that this is exactly the same as if you simply applied C^* to the above construction.

To this end let $J^{\mathcal{A}}_{\mathcal{B}} : \mathcal{A} \to \mathcal{B}$ be the inclusion map for $\mathcal{A}$ into $\mathcal{B}$ in $CA(X)$. Recall that $\varinjlim CA(X)$ is a quotient of $\bigoplus\{\mathcal{A} | \mathcal{A} \in CA(X)\}$ under the equivalence $f \sim g$ if for $f \in \mathcal{A}$ and $g \in \mathcal{B}$ there is some $\mathcal{D} \in CA(X)$ with $J^{\mathcal{A}}_{\mathcal{D}}(f) = J^{\mathcal{B}}_{\mathcal{D}}(g)$.

We will define $J : C^*(X) \to \varinjlim CA(X)$ by $[J(f)] = [f]$ (recall that $C^*(X) \in CA(X)$ so $C^*(X)$ embeds in $\varinjlim CA(X)$). Clearly this is an embedding (in fact, this is the "canonical" embedding of $C^*(X)$ into $\varinjlim CA(X)$). To show that this is surjective, let $[g] \in \varinjlim CA(X)$, then there is some $\mathcal{A} \in CA(X)$ and some $f \in \mathcal{A}$ so that f represents $[g]$. Now $\mathcal{A} \subset C^*(X)$ so $f \in C^*(X)$ which means that $J^{\mathcal{A}}_{C^*(X)}(f) = f$. However, this means that $J(f) = [f] = [g]$. Now J is a C^*-algebra homomorphism which is injective and surjective, so it is an isomorphism. Thus $C^*(X) = \varinjlim CA(X)$ as claimed.

The non-obvious fact that X is determined by the collection of all of its quotients is transformed by the functor C^* into the obvious fact that $C^*(X)$ is determined by the collection of all of its closed unital subalgebras.

Again, if all we know is the order structure of $CA(X)$ then we cannot use the direct limit

construction to recapture $C^*(X)$ (since the direct limit requires knowledge of the elements of each closed unital subalgebra, which we do not have in this case).

It is an amazing fact that the order structure of $P(X)$ is sufficient to determine X (this is just Magill's Theorem). Similarly amazing is the fact that the order structure of $CA(X)$ is sufficient to determine $C^*(X)$ (this is our Theorem 65).

Direct limit construction of X

We claim that $\varinjlim I(X) = X$. Again, if we define $\hat{I}(X) = I(X) \setminus \{X\}$, then $\hat{I}(X)$ is not a directed set so we cannot form $\varinjlim \hat{I}(X)$.

Define the function $J : X \to \varinjlim I(X)$ by $[J(x)] = [x]$. This is well defined since $\{x\} \in I(X)$ for each $x \in X$. (As an aside, contrast how easy it is to get the points of X in this case with how difficult it is to get the points of X in the case of $P(X)$). Clearly this is injective, since $[x] = [y]$ implies that there is some $A \in I(X)$ with $J_A^{\{x\}}(x) = J_A^{\{y\}}(y)$ but this cannot happen unless $x = y$ in X. The image of J is $\varinjlim I(X)$ since $J|_A : X \to A$ is surjective for each $A \in I(X)$. Thus J is a homeomorphism.

If all we know is the order structure of $I(X)$, then we still can reconstruct X. First, it is easy to get the points of X – they are simply all the elements of $I(X)$ which are immediately above the minimum element (which is $\emptyset$). Suppose we have a subset $A \subset X$, and we want to know what $cl(A)$ is. By definition, $cl(A) = \bigcap\{C | A \subset C\}$ where $C \subset X$ is closed. However, this is exactly the same thing as $\bigwedge\{C \in I(X) | \{x\} \leq C, \forall x \in A\}$. Thus, we can get the topology of X from the order structure.

Inverse limit construction of $C^*(X)$

Again, by "dualizing" this construction we can immediately see that $C^*(X) = \varprojlim PA(X)$. However, as before, we will explicitly show the details of the isomorphism. Again, $\widehat{PA}(X) = PA(X) \setminus \{C^*(X)\}$ is not a directed set so we cannot form $\varprojlim \widehat{PA}(X)$. Thus we will show that $C^*(X) = \varprojlim PA(X)$.

To this end let $\pi_{\mathcal{A}}^{\mathcal{B}} : \mathcal{B} \to \mathcal{A}$ be the canonical projection when $\mathcal{A} \leq \mathcal{B} \in PA(X)$. Define

$$\Pi : C^*(X) \to \varprojlim PA(X)$$

by

$$(\Pi(f))_{\mathcal{A}} = \pi_{\mathcal{A}}^{C^*(X)}(f).$$

To see that Π is well defined, just notice that

$$(\Pi(f))_A = \pi_A^{C^*(X)}(f) = \pi_A^B(\pi_B^{C^*(X)}(f)) = \pi_A^B((\Pi(f))_B)$$

so that $\Pi(f) \in \varprojlim PA(X)$.

Suppose $\Pi(f) = \Pi(g)$. This means that $\pi_{\mathcal{A}}^{C^*(X)}(f) = \pi_{\mathcal{A}}^{C^*(X)}(g)$ for all $\mathcal{A} \in PA(X)$. Let M and N be maximal ideals in $C^*(X)$. Then we get two different projections $p_1 : C^*(X) \to I\!R$ and $p_2 : C^*(X) \to I\!R$ so that $(p_1)^{-1}(0) = M$ and $(p_2)^{-1}(0) = N$. Thus, in particular, $p_1(f) = p_1(g)$ and $p_2(f) = p_2(g)$. Generalizing this, we get that $p(f) = p(g)$ for all projections arising from maximal ideals, which implies that $f = g$. Thus, Π is injective.

Choose $h \in \varprojlim PA(X)$, and let $f \in C^*(X)$ be such that $(\Pi(f))_{C^*(X)} = (h)_{C^*(X)}$. Then $(\Pi(f))_{\mathcal{A}} = \pi_{\mathcal{A}}^{C^*(X)}(f) = \pi_{\mathcal{A}}^{C^*(X)}(g) = (g)_{\mathcal{A}}$ for all $\mathcal{A} \in PA(X)$. Thus, $\Pi(f) = g$ so Π is surjective. Now Π is an algebra homomorphism which is bijective, so it is an isomorphism. Thus, $C^*(X)$ is isomorphic to $\varprojlim PA(X)$.

In this case, if all we know is the order structure of $PA(X)$, then it is easy to get the maximal ideals, since they are simply the elements of $PA(X)$ which are immediately above the minimum element of $PA(X)$ (which is the projection of $C^*(X)$ onto 0). Thus we have the maximal ideal space. To get the hull-kernel topology is also easy, since the hull-kernel topology is defined using only the order structure on the ideals and we know the order structure of the ideals ($\mathcal{I} \leq \mathcal{J}$ if and only if $p_{\mathcal{I}} \geq p_{\mathcal{J}}$ where $p_{\mathcal{I}}$ is the projection canonically associated with $\mathcal{I}$). So let $\mathcal{A}$ be a collection of maximal ideals, then

$$cl(\mathcal{A}) = \{M \in MI(X) | M \geq \bigwedge\{I | I \in \mathcal{I}\}\}.$$

This is clearly defined using only the order structure of $PA(X)$.

Remarks

Thus, there are a couple of ways to construct the space X – as an inverse limit or as a direct limit. For the direct limit, it is easy to recover the space X from just the order structure of the directed set. For the inverse limit, this is much more difficult.

These two constructions correspond to two constructions for $C^*(X)$ via the contravariant functor C^*. In this case, it is easy to recover $C^*(X)$ from just the order structure of the directed set for the inverse limit construction but it is much harder to reconstruct $C^*(X)$ from the order structure of the directed set in the direct limit construction.

As we previously mentioned, our results (as well as Magill's Theorem 57) can be interpreted in terms of the faithfulness of certain functors. More specifically, Magill's Theorem states that the functor from $CompHaus$ to PO which takes a compact Hausdorff space H and gives the lattice of Hausdorff projections of H (what we called $P(X)$) is a faithful functor (which means that two different spaces yield two different partially ordered sets). Similarly, our Theorem 65 states that the functor which takes a C^*-algebra $\mathcal{A}$ and gives the lattice of all closed unital subalgebras of $\mathcal{A}$ is a faithful functor.

Now we turn to a brief discussion of the "structure space" (or the space of maximal ideals with the hull-kernel topology). For a commutative C^*-algebra $\mathcal{A}$ this space contains enough information to classify $\mathcal{A}$ (since, in this case, $\mathcal{A} \cong C^*(X)$ for some compact Hausdorff space and the structure space is homeomorphic to X). Notice that what the structure space provides is the structure of the collection of maximal ideals – the points of the structure space are the maximal ideals and we get information about non-maximal ideas from the topology of the structure space. It is possible to define the structure space for a general ring. However, in general the structure space will not classify the ring. For example, let M_n be the collection of $n \times n$ real matrices. Then the only ideals of M_n are M_n and 0 – the trivial ideal. Thus the structure space will not distinguish between M_n and M_m.

Let $DA(X)$ be the collection of maximal proper closed unital subalgebras of $C^*(X)$. We can put some kind of "structure" on $DA(X)$ similar to the hull-kernel topology. Thus, for

$S \subset DA(X)$, let

$$\overline{S} = \{A \in DA(X) | A \geq \bigwedge S\}.$$

This "closure" operator completely captures the order structure of $CA(X)$. However, it is NOT a closure operator since $\overline{S \cup T} \neq \overline{S} \cup \overline{T}$. Thus, $DA(X)$ is not a topological space in any natural way. This is another reason why it is more difficult to prove that $CA(X)$ classifies $C^*(X)$ (or, equivalently, that $P(X)$ classifies X) than it is to prove that $PA(X)$ classifies $C^*(X)$. We call $DA(X)$ the "dual" structure space of $C^*(X)$, since in some sense it is a "dual" construction (defined from maximal injections rather than maximal projections).

4.6 Final Comments

In his 1966 paper [J], Jerison has a result which is similar to Lemma 60, (recall that Lemma 60 states that $C^*(X)/C_0(X)$ is sufficient to determine $\beta X \setminus X$ in that if $C^*(X)/C_0(X) \cong C^*(Y)/C_0(Y)$ then $\beta X \setminus X \cong \beta Y \setminus Y$ and conversely). If we let $C_c(X)$ be the functions in $C^*(X)$ with compact support (so that $C_c(X)$ is an ideal in $C^*(X)$), then Jerison's result states that for locally compact X and Y with $X = \bigcup_n K_n$ for compact K_n and similarly for Y, if

$$(C_0(X) \oplus 1)/C_c(X) \cong (C_0(Y) \oplus 1)/C_c(Y)$$

then

$$\beta X \setminus X \cong \beta Y \setminus Y.$$

It is relatively straightforward to see that

$$(C_0(X) \oplus 1)/C_c(X) \cong C_0(X)/C_c(X) \oplus 1$$

so it is enough to know $C_0(X)/C_c(X) \cong C_0(Y)/C_c(Y)$. Jerison uses the space of minimal prime ideals instead of the space of maximal ideals, thus is able to use $C_0(X)$ instead of $C^*(X)$.

A converse to this is not known, but would be very interesting. Functions in $C_c(X)$ are functions which are zero on every neighborhood of ∞ (where we are thinking of ∞ as the

added point in the one-point compactification of X). Thus, we can think of $C_0(X)/C_c(X)$ as giving some information about a neighborhood of ∞. A converse would thus state that $\beta X \setminus X$ "determines" the neighborhood of ∞. There are a few problems, however. Since $cl(C_c(X)) = C_0(X)$, and $C_0(X) \neq C_c(X)$ in general, we know that $C_c(X)$ is not a closed ideal. Thus, $C_0(X)/C_c(X)$ has no natural norm so is not a C^*-algebra (it is not even a normed vector space). Also, locally compact is not enough in order to get information about a neighborhood of ∞ from information about $\beta X \setminus X$ (or information about $C_0(X)/C_c(X)$). For example, if we let $X = W(\omega_1)$ and $Y = W(\omega_2)$ then $\beta X \setminus X = \{\infty\} = \beta Y \setminus Y$ but clearly no neighborhood of ∞ in X can be homeomorphic to any neighborhood of ∞ in Y (this is the same example we gave in the introduction to this chapter, right after Proposition 56). Notice that in this case, $C_0(X) = C_c(X)$ and likewise $C_0(Y) = C_c(Y)$. Also, this is cheating a little since $W(\omega_1)$ is not the union of countably many compact sets.

If X is a locally connected generalized continuum, then clearly X is the countable union of compact sets, so Jerison's result applies. Is the converse true for this class of spaces?

There are many other open questions in this area. Suppose that $\Gamma : K(X) \to K(Y)$ is just a lattice map, do we get a mapping between $C^*(X)/C_0(X)$ and $C^*(Y)/C_0(Y)$? Conversely, if $\Lambda : C^*(X)/C_0(X) \to C^*(Y)/C_0(Y)$ is an algebra homomorphism, do we get a lattice map $K(X) \to K(Y)$? Furthermore, if we restrict our attention to nicer classes of spaces, can we get sharper results? (See the comments at the end of Chapter 3 for more on this).

Are there any function-theoretic results which will tell us when $K(X) \cong K(Y)$ for arbitrary X and Y (*i.e.* not locally compact)?

Finally, is there some intrinsic way to use the "dual" structure space, $DA(X)$, to capture the properties of $C^*(X)$? Also, are there other uses of $DA(X)$? We know that $DA(X)$ is not a topological space, but does it have useful or interesting properties? Does its structure correspond to some known topological construction?

Chapter 5

Function Algebras, Completely Regular Topologies, and Partitions

The relationship between $K(X)$ and $PSA(X)$ leads one to inquire about the relationship between partitions of a space and closed unital algebras of functions on the same space. We turn now to this investigation.

5.1 Introduction

It is well-known that there is a strong relationship between algebras of real-valued functions on a topological space and the topology on the space. One example of this relationship is that a space is completely regular if and only if it has the weak topology by the algebra of all continuous bounded real-valued functions on the space. Another example is the fact that the lattice of closed unital subalgebras of $C^*(X)$ is isomorphic to the lattice of Hausdorff compactifications of X for a locally compact completely regular Hausdorff space X.

In this chapter, we explore this relationship for a finite space. We show that the collection of partitions of the space S is isomorphic to the set of algebras of real-valued functions on S. We also show that the collection of all algebras of real-valued functions on S is isomorphic to the collection of completely regular topologies on S. This will give us an easy characterization of all the completely regular topologies on S. Furthermore, it will make this relationship very explicit. A further result will be an enumeration of the homeomorphism classes of completely regular topologies on the finite set S. These results are not true for infinite spaces, and we will give some very easy arguments and examples to show why.

The relationship between algebras and topologies is easy to describe intuitively. Let τ be a completely regular topology on the finite set S. To τ we associate the algebra of all τ-continuous real-valued functions on S, call this algebra $\mathcal{A}_\tau$. As we mentioned above, $\mathcal{A}_\tau$ completely determines the topology on S. It is clear that if we have two different completely regular topologies on S, τ and σ, and if $\tau \subset \sigma$ then $\mathcal{A}_\tau \subset \mathcal{A}_\sigma$. Similarly if $\mathcal{A}_\tau \subset \mathcal{A}_\sigma$ then $\tau \subset \sigma$. This indicates that the lattice of all completely regular topologies on S is isomorphic to the lattice of all algebras of real-valued functions on S. When S is finite, every algebra of functions is closed (with any linear topology). This is not true if S is infinite, which is one reason the relationship is not as simple for infinite spaces.

The relationship between algebras and partitions is also easy to describe. Given a partition $\mathcal{P}$ of the finite set S, we let $\mathcal{A}_P = \{f \mid f|_T = \text{ constant } \forall T \in \mathcal{P}\}$. Then $\mathcal{A}_P$ is a closed algebra. Conversely, for any algebra $\mathcal{A}$, we get the partition of S given by the maximal stationary sets of $\mathcal{A}$. Again this correspondence is order preserving, so the lattice of all algebras of real-valued functions on S is isomorphic to the lattice of all partitions of S (ordered under refinement).

Using these two relationships, it is easy to get a relationship between the partitions of the set S and completely regular topologies on the set S. Given a partition $\mathcal{P}$ of S, we get a topology on S by first putting the discrete topology on $\mathcal{P}$ and then pulling back this topology to S by the natural quotient map. This always gives a completely regular topology. If S is finite, then this procedure gives all of the completely regular topologies on S (see Theorem 89). Conversely, given a completely regular topology on S, we get a partition of S where the elements of the partition are the closures of the points (this generates a partition since S is completely regular). In the finite case, this gives an isomorphism between the lattice of partitions of S and the lattice of completely regular topologies on S.

The situation is different in several ways when S is infinite. Many different algebras can give you the same partition. Furthermore, if we start with an algebra of functions $\mathcal{A}$ and generate a partition $\mathcal{P}$ from the maximal stationary sets of $\mathcal{A}$ and then use this partition to generate a new algebra $\mathcal{A}_P$, we have that $\mathcal{A}_P$ is a pointwise closed algebra which contains $\mathcal{A}$ even though there is no reason that $\mathcal{A}$ should be pointwise closed (when S is finite, $\mathcal{A}$ is

always the same as $\mathcal{A}_P$). Another difference is that if you start with two different algebras, you might get the same topology. However, if the topology you get is compact, then two different closed unital subalgebras will generate two different topologies (this is from the Gelfand Representation for Abelian C^*-algebras). If you start with a topology, you always get a uniformly closed algebra, but not necessarily a pointwise closed algebra (the uniform topology is the topology given by the norm $|f| = \sup_x |f(x)|$ and the pointwise topology is the topology given by $f_n \to f$ if $f_n(x) \to f(x)$ for all x). Finally, starting with different topologies can give you the same partition.

What makes the finite case so nice is the fact that all linear topologies on $\mathbb{R}^n$ are equivalent, the fact that every topology on a finite set S is compact, and the fact that the only Hausdorff topology on a finite set is the discrete topology. Thus, there is no difference between uniformly closed and pointwise closed. The fact that every topology on a finite set is compact implies that we can use an easy corollary of the Stone-Weierstrass Theorem to characterize topologies in terms of closed unital algebras. However, all finite dimensional algebras are closed, so we get a correspondence between algebras and topologies. Finally, the fact that the only Hausdorff topology on a finite set is the discrete topology lets us characterize topologies in terms of partitions using the idea of a Tychonoff quotient (defined in the next section).

5.2 Preliminaries

Throughout this chapter, S denotes a finite set. If $\mathcal{A}$ is a set of functions on S, a *stationary set* of $\mathcal{A}$ is a set $T \subset$S so that $f|_T$ is constant for all $f \in \mathcal{A}$. We will primarily be interested in the *maximal* stationary sets of $\mathcal{A}$. Note that the set of maximal stationary sets of the collection $\mathcal{A}$ forms a partition of S.

If $\mathcal{P}$ and $\mathcal{Q}$ are two partitions of S, we will say that $\mathcal{Q} \leq \mathcal{P}$ whenever $\mathcal{P}$ refines $\mathcal{Q}$. Clearly with this order, the collection of partitions of S forms a lattice.

If $\mathcal{F}$ is a collection of real-valued functions on a set S, then we denote by $alg(\mathcal{F})$ the unital algebra generated by $\mathcal{F}$; *i.e.*, $alg(\mathcal{F})$ is the smallest unital subalgebra of $\mathbb{R}^S$ which

contains $\mathcal{F}$ (recall that $I\!R^S$ is the algebra of all real-valued functions on S).

Let A and B be sets with a function $\phi : A \to B$. Then ϕ induces an algebra homomorphism $\phi^* : I\!R^B \to I\!R^A$ defined by $\phi^*(f)(a) = f(\phi(a))$ for all $f \in I\!R^B$. We call this the *pull-back* of ϕ. If ϕ is injective, then ϕ^* will be surjective and conversely.

In this chapter, completely regular does not imply Hausdorff (unlike in [GJ]). We are thus suspending our convention that all spaces be Hausdorff.

If X is a completely regular space, the *Tychonoff quotient* is the quotient space T defined by the equivalence relation $x \sim y$ when $f(x) = f(y)$ for all continuous bounded real-valued functions $f : X \to I\!R$. We put the quotient topology on the space T. Then we have that $C^*(T) \cong C^*(X)$ with the isomorphism given by q^*, where $q : X \to T$ is the quotient map. Notice that this implies that X has the weak topology by q.

We start by stating a theorem of Schur on determinants of partitioned matrices. We need this formula to compute determinants in the proof of Theorem 74.

Proposition 73 (Schur [CT]) *Let A, B, C, D be matrices of appropriate sizes with A invertible. Then*

$$det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = det(A)det(D - CA^{-1}B).$$

Proof: Just notice that

$$\begin{pmatrix} I & 0 \\ -CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & D - CA^{-1}B \end{pmatrix} \blacksquare$$

We will use a version of the Stone-Weierstrass Theorem. However, in our case, all the spaces will be finite. Thus, we prove the specific version we need.

Theorem 74 (Stone-Weierstrass) *Let $S = \{1, \ldots, n\}$ and let $\mathcal{F}$ be a non-empty collection of real-valued functions on S which separates the points of S. Then $alg(\mathcal{F}) = I\!R^n$.*

Proof: The proof is by induction on n.

If $n = 1$, then clearly the theorem is true.

Induction Step Suppose the theorem is true for all sets of cardinality n and all point separating collections of functions on such sets.

Let $\mathcal{F} = \{f_1, \ldots, f_m\}$ separate the points of $S = \{1, \ldots, n, n+1\}$. Then, since $\mathcal{F}$ separates the points of $\{1, \ldots, n\}$, by the induction hypothesis there are functions $g_1, \ldots, g_n \in alg(\mathcal{F})$ so that

$$g_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0, a_i)$$

where the 1 is in the ith position. Without loss of generality, we can assume that $a_i \geq 0$ for all $i = 1, \ldots, n$ (if not, simply replace g_i by g_i^2).

Let $d = \sum_{i=1}^{n} a_i$.

We divide the proof up into several cases. In each case, we will construct a basis of $I\!R^{n+1}$ from the g_i's and some other function in $alg(\mathcal{F})$.

Case 1 At least two of the a_i's are nonzero.

In this case, it is easy to guarantee that $d \neq 1$ (if $d = 1$, then simply replace one of the g_i's by g_i^2).

Consider the collection $\{g_1, \ldots, g_n, \mathbf{1}\}$, where by $\mathbf{1}$ we mean the constant function with value 1. Then by using Schur's formula (Proposition 73),we get

$$det \begin{pmatrix} g_1 \\ \vdots \\ g_n \\ \mathbf{1} \end{pmatrix} = det \begin{pmatrix} 1 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & \cdots & 0 & a_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & a_n \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix} = 1 - d \neq 0$$

Thus the set of functions $\{g_1, \ldots, g_n, \mathbf{1}\}$ is independent so this set spans $I\!R^{n+1}$.

Case 2 Only one of the a_i's is nonzero.

Without loss of generality, $a_n = 1$ (if $a_n \neq 1$, then the same argument as in Case 1 will show that the set of functions $\{g_1, \ldots, g_n, \mathbf{1}\}$ is independent). Notice that this is exactly the case where the functions $g_1, \ldots, g_n$ fail to separate the points $\{1, \ldots, n, n+1\}$. Since $\mathcal{F}$ separates the points of S, there is some $h \in \mathcal{F}$ so that $h(n) \neq h(n+1)$. Replacing h by $h + c$ if necessary, we can assume that $h \geq 0$. Then, again using Schur's formula,

$$det\begin{pmatrix} g_1 \\ \vdots \\ g_n + h \\ \mathbf{1} \end{pmatrix} = det\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h(1) & h(2) & \cdots & h(n)+1 & h(n+1)+1 \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix} =$$

$$(h(n)+1)\left(1 - \frac{h(n+1)+1}{h(n)+1}\right) \neq 0$$

since $h(n+1)+1 \neq h(n)+1$. Thus, we have that the set of functions $\{g_1, \ldots, g_{n-1}, g_n + h, \mathbf{1}\}$ is linearly independent; so these functions span $I\!R^{n+1}$. ∎

We will actually need a slight generalization of this, so we prove it next.

Corollary 75 *Let $S = \{1, \ldots, n\}$ and let $\mathcal{F}$ be a collection of real-valued functions on S. Then $alg(\mathcal{F})$ is the set of all functions on S which are constant on the stationary sets of $\mathcal{F}$.*

Proof: Let $\mathcal{P}$ be the partition of S induced by the maximal stationary sets of $\mathcal{F}$ and let $q : S \to \mathcal{P}$ be the natural quotient map. Then each $f \in \mathcal{F}$ descends to a well-defined function $f_\#$ on $\mathcal{P}$ since f is constant on each element of $\mathcal{P}$. By definition the collection $\mathcal{F}_\# = \{f_\# |\ f \in \mathcal{F}\}$ separates the points of $\mathcal{P}$, thus $alg(\mathcal{F}_\#) = I\!R^{\mathcal{P}}$ by the preceeding theorem. The quotient map q is surjective, so q^* is injective. Since q^* is an injective algebra homomorphism and $\mathcal{F} \subset q^*(alg(\mathcal{F}_\#))$, we know $alg(\mathcal{F}) = q^*(alg(\mathcal{F}_\#))$. Clearly $q^*(I\!R^{\mathcal{P}})$ is the set of all functions on S which are constant on the elements of $\mathcal{P}$. ∎

5.3 Main Results

5.3.1 Topologies and Algebras

We will begin by discussing the relationship between completely regular topologies on S and algebras of real-valued functions on S. It is well-known that completely regular topologies are precisely those topologies for which the bounded real-valued continuous functions generate the topology. Another way of looking at this is given a collection $\mathcal{F}$ of bounded real-valued functions on S, we get a completely regular topology on S by taking the weak topology by $\mathcal{F}$.

Definition 17 *Let $\mathcal{F}$ be a collection of real-valued functions on a set S. Then we denote by $\tau_{\mathcal{F}}$ the weak topology generated by $\mathcal{F}$ and call this* the topology generated by $\mathcal{F}$.

So given a collection of functions, we generate a completely regular topology. Our goal now will be to understand this process. Two different collections of functions might generate the same topology. The following propositions indicate when this happens.

Proposition 76 *Let $\mathcal{F}$ be a collection of real-valued functions on a finite set S. Let $\tau_{\mathcal{F}}$ be the topology generated by $\mathcal{F}$. Then the set of all $\tau_{\mathcal{F}}$-continuous real-valued functions is $alg(\mathcal{F})$.*

Proof: Let $\mathcal{A}$ be the set of all $\tau_{\mathcal{F}}$-continuous real-valued functions on S. Clearly $\mathcal{F} \subset \mathcal{A}$ and $\mathcal{A}$ is an algebra of functions (since the sum and product of continuous functions is continuous) so $alg(\mathcal{F}) \subset \mathcal{A}$. Suppose $alg(\mathcal{F}) \neq \mathcal{A}$. By Corollary 75, $alg(\mathcal{F})$ is the set of all functions which are constant on the stationary sets of $\mathcal{F}$. Thus, there is some $h \in \mathcal{A}$ and $x, y \in S$ with $h(x) \neq h(y)$ but $f(x) = f(y)$ for all $f \in alg(\mathcal{F})$. Since h is continuous, there is an $O \subset S$ with O $\tau_{\mathcal{F}}$-open and $x \in O$ but $y \notin O$. Without loss of generality, we assume that O is a basic open set so that there are functions $f_i \in \mathcal{F}$ and open subsets $U_i \subset I\!R$, with $i = 1, \ldots, n$, such that $O = f_1^{-1}(U_1) \bigcap \ldots \bigcap f_n^{-1}(U_n)$. But then, $x \in O$ implies $f_i(x) \in U_i$ for $i = 1, \ldots, n$ while $y \notin O$ implies there is some $j \in \{1, \ldots, n\}$ so that $f_j(y) \notin U_j$ while

$f_j(x) \in U_j$. This contradicts the fact that $f_j(x) = f_j(y)$, which is true since $f_j \in \mathcal{F}$. Therefore, $\mathcal{A} = alg(\mathcal{F})$. ▮

Proposition 77 *Let $\mathcal{F}$ and $\mathcal{G}$ be two collections of real-valued functions on a finite set S. Then $\mathcal{F}$ and $\mathcal{G}$ generate the same topology if and only if $\mathcal{F}$ and $\mathcal{G}$ generate the same algebra of functions.*

Proof: Suppose that $\tau_{\mathcal{F}} = \tau_{\mathcal{G}}$. Then the set of $\tau_{\mathcal{F}}$-continuous real-valued functions is the same as the set of $\tau_{\mathcal{G}}$-continuous real-valued functions. By the previous proposition, $alg(\mathcal{F}) = alg(\mathcal{G})$. Conversely, suppose that $alg(\mathcal{F}) = alg(\mathcal{G})$. Then by the previous proposition, the set of $\tau_{\mathcal{F}}$-continuous real-valued functions is the same as set of $\tau_{\mathcal{G}}$-continuous real-valued functions. However, $\tau_{\mathcal{F}}$ is the weak topology by $alg(\mathcal{F}) = alg(\mathcal{G})$ and $\tau_{\mathcal{G}}$ is the weak topology by $alg(\mathcal{G}) = alg(\mathcal{F})$. Thus, $\tau_{\mathcal{F}} = \tau_{\mathcal{G}}$. ▮

We introduce some notation in order to streamline the discussion.

Notation 18 *$Top(S)$ is the lattice of completely regular topologies on S.*

$Part(S)$ is the lattice of partitions on S.

$Alg(S)$ is the lattice of unital algebras of real-valued functions on S.

Theorem 78 *$Alg(S)$ is lattice isomorphic to $Top(S)$.*

Proof: The isomorphism is $\Psi : Top(S) \to Alg(S)$ given by $\Psi(\tau) = \mathcal{A}_\tau$. By the definition of $\mathcal{A}_\tau$, if $\sigma, \tau \in Top(S)$ with $\sigma \subset \tau$ then $\mathcal{A}_\sigma \subset \mathcal{A}_\tau$ so Ψ is order-preserving. Proposition 77 shows that Ψ is injective. If $\mathcal{A} \in Alg(S)$, then by Proposition 76 the algebra of all $\tau_{\mathcal{A}}$-continuous functions is $alg(\mathcal{A}) = \mathcal{A}$. Thus, $\Psi(\tau_{\mathcal{A}}) = \mathcal{A}$ or Ψ is surjective. ▮

Notice that the function $\mathcal{A} \mapsto \tau_{\mathcal{A}}$ is the inverse for the function Ψ (given by $\tau \mapsto \mathcal{A}_\tau$) from the proof. However, this is only true when S is finite. The function Ψ is always injective and order-preserving. However, it is not surjective when S is infinite.

Example Let $I\!N = \{1, 2, 3, \ldots\}$ with the discrete topology, which we denote by τ. Then $\mathcal{A}_\tau = C^*(I\!N)$, call this $\mathcal{A}$. Let $\mathcal{B} = \{f \in C^*(I\!N) |\ \lim_{n\to\infty} f(n) \text{ exists } \} = C_0(I\!N) \oplus 1$; then $\mathcal{B}$ is a closed unital algebra of real-valued functions on $I\!N$ and $\tau_{\mathcal{A}} = \tau_{\mathcal{B}}$. In fact, for any closed unital subalgebra $\mathcal{I} \subset C^*(I\!N)$ such that $\mathcal{I}$ separates the points from closed sets of $I\!N$, we have $\tau_{\mathcal{I}} = \tau_{\mathcal{A}}$. This shows that the function Ψ is far from surjective.

The situation in the example is typical, as the following proposition shows.

Proposition 79 *Let X be a completely regular space and T be its Tychonoff quotient, with quotient map $q : X \to T$. Suppose $\mathcal{A} \subset C^*(T)$ with $\mathcal{A}$ a uniformly closed unital algebra which separates points from closed sets. Then $q^*(\mathcal{A})$ generates the topology of X.*

Proof: It suffices to show that $\mathcal{A}$ generates the topology of T, since q is a quotient map. Because T is completely regular, $\mathcal{A} = C^*(\alpha T)$ for some compactification αT of T. Thus, $\tau_{\mathcal{A}}$ is the topology on αT. Since T is embedded in αT we know that $\tau_{\mathcal{A}}$ restricts to the topology of T, or $\mathcal{A}$ generates the topology of T. ∎

Thus the function $\mathcal{A} \mapsto \tau_{\mathcal{A}}$ is not injective when S is infinite; it is not an inverse of Ψ from Theorem 78. There is a partial result similar to Proposition 77 for infinite sets.

Proposition 80 *Let T be an infinite set and $\mathcal{F}$ and $\mathcal{G}$ be collections of bounded real-valued functions on T. If $alg(\mathcal{F}) = alg(\mathcal{G})$, then $\tau_{\mathcal{F}} = \tau_{\mathcal{G}}$.*

Proof: This is an easy consequence of the fact that the sum and product of continuous real-valued functions are continuous. ∎

If we assume that the topology on T is compact, then it is true that $\mathcal{A} \mapsto \tau_{\mathcal{A}}$ is injective when we restrict $\mathcal{A}$ to be closed and unital. However, it is not an easy thing to tell when $\tau_{\mathcal{A}}$ is compact given $\mathcal{A}$ (in the terminology from [GJ], this occurs whenever every maximal ideal in $\mathcal{A}$ is fixed).

5.3.2 Partitions and Algebras

We now turn to the relationship between partitions on a set and algebras of real-valued functions on the same set. In the absence of any topology on the set or any continuity requirements on the functions, the relationship is very simple. If $\mathcal{A}$ and $\mathcal{B}$ are two different collections of functions then $\mathcal{A}$ and $\mathcal{B}$ can generate the same partition of S. In order to get an isomorphism between algebras of functions and partitions, it is necessary to know when $\mathcal{A}$ and $\mathcal{B}$ generate the same partition.

Proposition 81 *Let $\mathcal{F}$ be a collection of real-valued functions on a set X. Then the smallest pointwise closed unital algebra containing $\mathcal{F}$ is the algebra of all functions which are constant on the stationary sets of $\mathcal{F}$.*

Proof: Let $\mathcal{A}$ denote the algebra of all functions which are constant on the stationary sets of $\mathcal{F}$. We must show that every function in $\mathcal{A}$ is the pointwise limit of functions in $alg(\mathcal{F})$. Let T be the quotient of X under the equivalence relation $x \sim y$ when $f(x) = f(y)$ for $f \in \mathcal{F}$, so that $\mathcal{F}$ separates the points of T. By definition of T, $\mathcal{A}$ is algebra isomorphic to $\mathbb{R}^T$. Let $g \in \mathcal{A}$ and $\Lambda = \{D \subset T |\ D \text{ is finite}\}$ ordered by inclusion. Then Λ is directed set. Since $\mathcal{F}$ separates the points of T, by Theorem 74 for each $D \in \Lambda$ there is an $f_D \in alg(\mathcal{F})$ so that $f_D|_D = g|_D$. Let $t \in T$, then for any $D \in \Lambda$ so that $t \in D$ we have $f_D(t) = g(t)$, or f_D converges to g pointwise. ∎

Proposition 82 *Two collections of functions on T generate the same partition of T if and only if they generate the same pointwise closed unital algebra.*

Proof: Let $\mathcal{F}$ and $\mathcal{G}$ be two collections of functions on T. Suppose that $\mathcal{F}$ and $\mathcal{G}$ generate the same partition $\mathcal{P}$ of T, then their stationary sets are the same. By Proposition 81 they generate the same pointwise closed unital algebra. Conversely, suppose they generate the same pointwise closed unital algebra. Then again by Proposition 81 their stationary sets are the same; thus they generate the same partition. ∎

This proposition shows that the right collection of algebras to consider is the collection of pointwise closed algebras.

Theorem 83 *Let T be a set. Then there is a lattice isomorphism between $Part(T)$ and the lattice of all pointwise closed algebras in $Alg(T)$.*

Proof: The isomorphism is given by $\Phi(\mathcal{P}) = \mathcal{A}_P$, where $\mathcal{A}_P = \{f \mid f|_H = \text{ constant } \forall H \in \mathcal{P}\}$. Clearly $\mathcal{A}_P$ is an algebra. Since the pointwise limit of constant functions is constant, $\mathcal{A}_P$ is pointwise closed. If $\mathcal{P} \leq \mathcal{Q}$, (ie, $\mathcal{Q}$ refines $\mathcal{P}$), then clearly $\mathcal{A}_P \subset \mathcal{A}_Q$ so Φ is order-preserving. By the definition of $\mathcal{A}_P$, $H \in \mathcal{P}$ if and only if $f|_H$ is constant for all $f \in \mathcal{A}$ and if $H \subset G$ with $f|_G$ is constant for all $f \in \mathcal{A}$, then $H = G$, *i.e.* H is a maximal stationary set of $\mathcal{A}$. Therefore, $\mathcal{A}_P$ completely determines $\mathcal{P}$ so Φ is injective. Let $\mathcal{A}$ be a pointwise closed unital algebra in $Alg(T)$. By Proposition 81, $\mathcal{A}$ is the set of all functions which are constant on the stationary sets of $\mathcal{A}$. If we let $\mathcal{P} = \{H|\ H \subset T,\ H \text{ is a stationary set of } \mathcal{A}\}$, then $\mathcal{A} = \mathcal{A}_P$, so Φ is surjective. ∎

The inverse of Φ is the function $\mathcal{A} \mapsto \mathcal{P}_A = \{$ maximal stationary sets of $\mathcal{A}\}$.

There is no mention of finite or infinite in any of these results, so they hold in the general case. In the finite case, we have the special situation that all algebras are pointwise closed, so we get an easy corollary to Theorem 83.

Corollary 84 *For a finite set S, there is an isomorphism between $Alg(S)$ and $Part(S)$.*

Proof: Every finite dimensional algebra is pointwise closed. ∎

Why is it necessary to consider only unital algebras? Is $Part(S)$ isomorphic to the collection of algebras on S (not necessarily unital)? The answer is no. The following proposition helps to explain why.

Proposition 85 *If $\mathcal{A}$ is an algebra of real-valued functions on the set S, then there is a set T and a surjection $\pi : S \to T$ so that one of the following is valid:*

1. *$\mathcal{A}$ is an ideal.*

2. *If $\mathcal{A}$ contains the constants, then $\mathcal{A} = \pi^*(\mathbb{R}^T)$*

3. *$\mathcal{A} = \pi^*(\mathcal{I})$ for some ideal $\mathcal{I} \subset \mathbb{R}^T$.*

Proof: (Actually, type 1 and type 3 are the same, if we let the $S = T$ and $\pi = id_S$). First, we need the set T. This set is just the collection of all maximal stationary sets of $\mathcal{A}$ (*i.e.,* the partition induced by $\mathcal{A}$).

Clearly all these are subalgebras, since π^* is an algebra homomorphism. The only thing left to show is that if $\mathcal{A}$ is not of type 1 or 2, then it is of type 3.

In this case, $1 \notin \mathcal{A}$. Let $\mathcal{C} = \mathcal{A} \oplus 1 \subset \mathbb{R}^S$. We know by corollary 75 that $\mathcal{C}$ can be thought of as $\mathbb{R}^T$, thus it is of type 2. Furthermore, we can easily see that $\mathcal{A}$ is an ideal in $\mathcal{C}$, being the kernel of the homomorphism $\phi : \mathcal{C} \to \mathbb{R}$ defined by $\phi(f + c) = c$ where $f \in \mathcal{A}$ and $c \in \mathbb{R}$. Thus, as a subalgebra of $\mathbb{R}^T$, we know that $\mathcal{A} \cong \{f \in \mathbb{R}^T | f(p) = 0\}$ for some $p \in T$. This implies that $\mathcal{A} = \pi^*(\mathcal{I})$ where $\mathcal{I}$ is just the ideal $\{f \in \mathbb{R}^T | f(p) = 0\}$. ∎

So, what is the relationship between partitions of S and algebras of real-valued functions on S? Let $\mathcal{P}$ be a partition of S. Then by the preceeding proposition, there are many algebras of functions which induce $\mathcal{P}$. Take any maximal ideal $\mathcal{I} \subset \mathbb{R}^{\mathcal{P}}$ and consider $\pi^*(\mathcal{I}) \subset \mathbb{R}^S$; this will be an algebra in $\mathbb{R}^S$. However, only one of these algebras will be unital, the one corresponding to $\mathbb{R}^{\mathcal{P}}$. This is why we need the algebra to be unital to get an isomorphism.

When T is a compact Hausdorff space, there are some strikingly similar results. The proofs of these results are almost identical to the proofs in the general case. For comparison, we include them here. Notice that for a set of continuous functions the stationary sets are all closed.

Theorem 86 *Let $\mathcal{F}$ be a collection of bounded continuous real-valued functions on T. Then*

the smallest uniformly closed unital algebra containing $\mathcal{F}$ is the algebra of all continuous bounded functions which are constant on the stationary sets of $\mathcal{F}$.

Proof: The proof follows from the Stone-Weierstrass Theorem in exactly the same way as we proved Corollary 75 from Theorem 74. ∎

Theorem 87 *There is a lattice isomorphism between the lattice of all partitions of T induced by Hausdorff quotients of T and the lattice of all uniformly closed unital subalgebras of $C^*(T)$.*

Proof: The isomorphism is the function $\mathcal{P} \mapsto \mathcal{A}_P$, where here $\mathcal{A}_P = \{f \in C^*(T) \mid f|_H = \text{ constant } \forall H \in \mathcal{P}\}$ (similar to the proof of Theorem 83). Clearly this function preserves order and is injective. Let $\mathcal{A}$ be a uniformly closed unital subalgebra of $C^*(T)$. Let Y be the quotient of T under the relation $x \sim y$ if $f(x) = f(y)$ for all $f \in \mathcal{A}$, with quotient map q. Then Y is a Hausdorff quotient of T. Let $\mathcal{P}$ be the partition of T induced by q. Then $\mathcal{A}_P = \mathcal{A}$ by Theorem 86, so the function is surjective. ∎

Again, notice that we require the subalgebras to be unital. The reason is exactly the same as in the finite case, as the following theorem shows.

Theorem 88 *Let $\mathcal{A}$ be a closed subalgebra of $C^*(X)$ with X a compact Hausdorff space. Then there is a compact Hausdorff space T and a quotient map $\pi : X \to T$ so that one of the following holds:*

1. *$\mathcal{A}$ is an ideal.*

   or

2. *If $\mathcal{A}$ contains the constants, then $\mathcal{A} = \pi^*(C^*(T))$*

   otherwise

3. *$\mathcal{A} = \pi^*(\mathcal{I})$ for some closed ideal $\mathcal{I} \subset C^*(T)$.*

Proof: The proof of this theorem is almost identical to the proof of Proposition 85. However, we give it just to show the parallels.

(Again, type 1 and type 3 are the same). First, we need to define the set T. Just like Proposition 85, we define T to be the partition induced by the maximal stationary sets of $\mathcal{A}$. This is compact and Hausdorff since X is compact and Hausdorff.

Suppose that $\mathcal{A}$ is not of type 1 or 2. What we need to show is that it is of type 3. The proof of this is exactly the same as in Proposition 85, with the exception that now we need to show that the algebra $\mathcal{C} = \mathcal{A} \oplus 1$ is closed for it to work (we need this in order for $\mathcal{C}$ to be a C^*-algebra).

There is only one C^*-algebra norm on any unital *-algebra, so we have a unique choice of norm for $\mathcal{C} = \mathcal{A} \oplus 1$. In fact, this norm will extend the norm on $\mathcal{A}$. Define the homomorphism $\phi : \mathcal{C} \to I\!R$ by $\phi(f + c) = c$ where $f \in \mathcal{C}$ and $c \in I\!R$. Then clearly $\mathcal{A} \subset \mathcal{C}$ is the kernel of ϕ so $\mathcal{A}$ is a closed ideal in $\mathcal{C}$ (in fact, $\mathcal{A}$ is a maximal ideal in $\mathcal{C}$). Thus, $\mathcal{C}$ must be closed.

Now that we know that $\mathcal{C}$ is closed, we know that $\mathcal{C} \cong C^*(T)$ (in fact by Theorem 86, $\mathcal{C}$ is exactly the set of functions which are constant on elements of T). Thus, there is some $p \in T$ so that $\mathcal{A} = \{f \in C^*(X) | f|_p = 0\}$. This is the same thing as $\mathcal{A} = \pi^*(\mathcal{I})$ with $\mathcal{I} = \{f \in C^*(T) | f(p) = 0\}$. ∎

So again the problem is that there are many closed algebras for each partition, but there is only one closed unital algebra for each partition.

Notice that this is the same theorem as Theorem 67 in chapter 4. However, for ease of exposition we included it in both places.

5.3.3 Topologies and Partitions

Now we discuss the relationship between completely regular topologies on S and partitions on S. The relationship is significantly different for finite sets and infinite sets, unlike the relationship between algebras and partitions. First we discuss a method of constructing completely regular topologies in general.

Let T be a set and X be any quotient of T, with quotient map $q : T \to X$. If we endow X with a completely regular Hausdorff topology, this topology induces a completely regular topology on X via q – specifically, the weak topology by q. Any completely regular topology on T can be obtained this way, since we can take X to be the Tychonoff quotient of T.

Notice that when T is infinite, X can be infinite so there can be many different completely regular Hausdorff topologies on X. Each of these will induce a different topology on T, and all of these topologies will be unrelated (in the order on the lattice of all topologies on T). However, each of these topologies will generate the same partition of T.

This is not the case when T is finite. When T is finite, there is only one Hausdorff topology on X. This fact is why there is an isomorphism between $Part(T)$ and $Top(T)$.

Theorem 89 *$Part(S)$ is isomorphic to $Top(S)$ for a finite set S.*

Proof: Let $\mathcal{P}$ be a partition of S and $q : S \to \mathcal{P}$ be the natural quotient map. Give $\mathcal{P}$ the discrete topology and then let τ_P be the weak topology on S by q. With τ_P, S is completely regular. We claim that the function $\Upsilon : Part(S) \to Top(S)$ given by $\Upsilon(\mathcal{P}) = \tau_P$ is the required isomorphism. If $\mathcal{P}, \mathcal{Q} \in Part(S)$ with $\mathcal{P} \leq \mathcal{Q}$, then $q_Q \circ q_P = q_Q$ so $\tau_P \subset \tau_Q$ or Υ preserves order. If $\tau_P = \tau_Q$ then $q_P = q_Q$ so $\mathcal{P} = \mathcal{Q}$ and Υ is injective. Finally, let τ be a completely regular topology on S and let T be its Tychonoff quotient, with quotient map $q : S \to T$. Since T is Hausdorff and finite, T is discrete. Furthermore, τ is the weak topology by q. If we let $\mathcal{P} = \{q^{-1}(t) | t \in T\}$, then $\tau = \tau_P$ or $\Upsilon(\mathcal{P}) = \tau$. ∎

Another way of getting an isomorphism between $Part(S)$ and $Top(S)$ is via Corollary 84 and Theorem 78. Corollary 84 says that $Part(S)$ is isomorphic to $Alg(S)$ and Theorem 78 says that $Alg(S)$ is isomorphic to $Top(S)$. Combining these two isomorphisms, we get an isomorphism $\Upsilon^{\#} : Part(S) \to Top(S)$. Is $\Upsilon^{\#} = \Upsilon$? The answer is yes, as is easily shown.

Proposition 90 $\Upsilon^{\#} = \Upsilon$

Proof: We simply trace the two isomorphisms in the definition of $\Upsilon^{\#}$. From Theorem 83 we get the function $\Phi : Part(S) \to Alg(S)$ given by $\Phi(\mathcal{P}) = \mathcal{A}_P = \{f \mid f|_H =$

constant $\forall H \in \mathcal{P}\}$. From Theorem 78 we get $\Psi^{-1} : Alg(S) \to Top(S)$ given by $\Psi^{-1}(\mathcal{A}) = \tau_{\mathcal{A}}$. Let $\mathcal{P}$ be a partition with quotient map $q : S \to \mathcal{P}$. Then $\Phi(\mathcal{P}) = \mathcal{A}_P$ is the set of all functions which are constant on each element of $\mathcal{P}$ and $\Psi^{-1}(\mathcal{A}_P) = \tau_{\mathcal{A}_P}$. We will be done if we can show that $\tau_{\mathcal{A}_P}$ is the weak topology by q when $\mathcal{P}$ has the discrete topology. We do this by showing that $\mathcal{P}$ with the discrete topology is the Tychonoff quotient of S, when S has the topology $\tau_{\mathcal{A}_P}$. By definition, every $f \in \mathcal{A}_P$ is constant on each element of $\mathcal{P}$. Furthermore, by Proposition 76, $\mathcal{A}_P$ is the set of all $\tau_{\mathcal{A}_P}$-continuous functions. Thus $\mathcal{P}$ is the partition induced by the equivalence relation $x \sim y$ when $f(x) = f(y)$ for all $\tau_{\mathcal{A}_P}$-continuous $f : S \to I\!R$. Since $\mathcal{A}_P$ separates the points of $\mathcal{P}$, this quotient will get the discrete topology. This means that $\mathcal{P}$ is the Tychonoff quotient of S. ∎

So in the finite case, all of the above isomorphisms are compatible.

We can give a simple construction of the inverse of Υ. Let $\tau \in Top(S)$ and $s \in S$. Define $C(s,\tau) = \bigcap\{C|\ s \in C,\ C \text{ is closed }\}$. Then either $C(s,\tau) = C(\tilde{s},\tau)$ or $C(s,\tau) \bigcap C(\tilde{s},\tau) = \emptyset$, so the $C(s,\tau)$'s form a partition of S - call this partition $\mathcal{P}_\tau$. The desired inverse of Υ is the function $\tau \mapsto \mathcal{P}_\tau$.

One consequence of Theorem 89 is a characterization of when two completely regular topologies on a finite set are homeomorphic in terms of the partitions which are associated with them. The following proof is in the spirit of the isomorphism $\Upsilon^{\#} : Part(S) \to Top(S)$.

Theorem 91 *Two completely regular topologies on a finite set are homeomorphic if and only if the partitions associated with them are permutations of each other.*

Proof: By Theorem 89, the collection of all completely regular topologies on S is bijective with the collection of all partitions of S. Let $\mathcal{P}$ and $\mathcal{Q}$ be two partitions of S such that there is some permutation $\pi : S \to S$ with $\pi(\mathcal{P}) = \mathcal{Q}$. Let $\mathcal{A}$ be the algebra of all real-valued functions on S which are stationary on every element of $\mathcal{P}$. Define $\mathcal{B}$ similarly from $\mathcal{Q}$.

Let $f \in \mathcal{B}$ and $E \in \mathcal{P}$. Since $\pi(E) \in \mathcal{Q}$ and $f \in \mathcal{B}$, we see $f|_{\pi(E)}$ is constant. Thus, $\pi^*(f) \in \mathcal{A}$. The function π^* is an algebra homomorphism and is bijective since π is. Therefore, π^* is an isomorphism between $\mathcal{A}$ and $\mathcal{B}$.

Using this fact, we now show that π is a homeomorphism from S with the $\tau_{\mathcal{A}}$ topology to S with the $\tau_{\mathcal{B}}$ topology. Clearly π is bijective, so we need only show that π is continuous with a continuous inverse. To show that π is continuous, let O be a $\tau_{\mathcal{B}}$ open set of the form $O = f^{-1}(U)$ for some $f \in \mathcal{B}$ and an open $U \subset I\!R$. However, then

$$\pi^{-1}(O) = \pi^{-1}(f^{-1}(U)) = (f \circ \pi)^{-1}(U) = (\pi^*(f))^{-1}(U)$$

is a $\tau_{\mathcal{A}}$ open set, since $\pi^*(f) \in \mathcal{A}$. If we let O be an arbitrary basic $\tau_{\mathcal{B}}$ open set, then $O = f_1^{-1}(U_1) \bigcap \ldots \bigcap f_n^{-1}(U_n)$ for some $f_i \in \mathcal{B}$ and some open $U_i \subset I\!R$, for $n = 1, \ldots, n$. However, since π is a bijection, we have

$$\begin{aligned} \pi^{-1}(O) &= \pi^{-1}\Big(f_1^{-1}(U_1) \bigcap \ldots \bigcap f_n^{-1}(U_n)\Big) \\ &= \pi^{-1}(f_1^{-1}(U_1)) \bigcap \ldots \bigcap \pi^{-1}(f_n^{-1}(U_n)) \\ &= (\pi^*(f_1))^{-1}(U_1) \bigcap \ldots \bigcap (\pi^*(f_n))^{-1}(U_n) \end{aligned}$$

which is a $\tau_{\mathcal{A}}$ open set. Thus π is a continuous map.

The fact that π^{-1} is continuous is proved exactly the same way, using π^{-1} in place of π. Thus, π is a homeomorphism, as claimed.

Conversely, suppose that $\tau_{\mathcal{A}}$ and $\tau_{\mathcal{B}}$ are two completely regular topologies on S with generating algebras $\mathcal{A}$ and $\mathcal{B}$ respectively. Further suppose that there is a homeomorphism π from $(S, \tau_{\mathcal{A}})$ to $(S, \tau_{\mathcal{B}})$. Clearly π is a permutation of S. As before, we get an isomorphism $\pi^* : \mathcal{B} \to \mathcal{A}$ defined as $\pi^*(f)(s) = f(\pi(s))$. Let $\mathcal{P}$ be the partition of S induced by the maximal stationary sets of $\mathcal{A}$ and similarly $\mathcal{Q}$ induced by $\mathcal{B}$. We want to show that $\pi(\mathcal{P}) = \mathcal{Q}$. To this end, let $E \in \mathcal{P}$ and $f \in \mathcal{B}$. Then

$$f|_{\pi(E)} = (f \circ \pi)|_E = \pi^*(f)|_E$$

is constant since $\pi^*(f) \in \mathcal{A}$ and $E \in \mathcal{P}$. Since this is true for all $f \in \mathcal{B}$, it must be the case that $\pi(E) \in \mathcal{Q}$. Thus, $\pi(\mathcal{P}) = \mathcal{Q}$. ∎

We give another proof more in the spirit of the isomorphism $\Upsilon : Part(S) \to Top(S)$.

First we prove a lemma which is interesting in its own right.

Lemma 92 *Let $\mathcal{P} \in Part(S)$ and $\tau_P = \Upsilon(\mathcal{P})$, where Υ is the isomorphism from Theorem 89. Let $q : S \to \mathcal{P}$ be the quotient map. Then for each $s \in S$ with $q(x) = t$, $C(s, \tau_P) = q^{-1}(t)$. Thus, the partition induced by the $C(s, \tau_P)$'s is the same as $\mathcal{P}$.*

Proof: Since $q^{-1}(t)$ is closed and $q(s) = t$, then clearly $C(s, \tau_P) \subset q^{-1}(t)$. Suppose that $C(s, \tau_P) \neq q^{-1}(t)$, then there is some $x \in S$ and U a basic τ_P-open set so that $q(x) = t$ and $x \in U$ with $U \cap q^{-1}(t) = \emptyset$. Since U is a basic τ_P-open set, $U = q^{-1}(u)$ for some $u \in \mathcal{P}$. However, then $u = t$ since $q^{-1}(u) \cap q^{-1}(t) = \emptyset$ if $u \neq t$. This is a contradiction since $C(s, \tau_P) \subset q^{-1}(t) = q^{-1}(u) = U$. ∎

Now the alternate proof of Theorem 91.

Proof: Like the last proof, what we show is that two completely regular topologies are homeomorphic if and only if the partitions associated with them (by the isomorphism from Theorem 89) are permutations of each other. Let $\mathcal{P}, \mathcal{Q} \in Part(S)$ and $\tau_P = \Upsilon(\mathcal{P})$ and $\tau_Q = \Upsilon(\mathcal{Q})$. It suffices to prove that (S,τ_P) is homeomorphic to (S,τ_Q) if and only if there is some permutation $\pi : S \to S$ with $\pi(\mathcal{P}) = \mathcal{Q}$.

Suppose that $\pi : (S, \tau_P) \to (S, \tau_Q)$ is a homeomorphism. Then $\pi(C(s, \tau_P)) = C(\tilde{s}, \tau_Q)$ for some $\tilde{s} \in S$ (since C(s,τ) is defined only using τ and π is a homeomorphism). By Lemma 92, this is the same thing as $\pi(\mathcal{P}) = \mathcal{Q}$.

Conversely, suppose that $\pi : S \to S$ is a permutation with $\pi(\mathcal{P}) = \mathcal{Q}$. Then by Lemma 92, we know $\pi(C(s, \tau_P)) = C(\tilde{s}, \tau_Q)$ for some $\tilde{s} \in S$. Since the set of $C(s, \tau_P)$'s is a basis for (S, τ_P) and the set of $C(s, \tau_Q)$'s is a basis for (S, τ_Q) this implies that π is a homeomorphism. ∎

As a simple corollary to Theorem 91, we get an enumeration of all completely regular topologies on a finite set.

Corollary 93 *The number of homeomorphism classes of completely regular topologies on a finite set of cardinality n is* part(n)*, the number of partitions of the integer n.*

Proof: Theorem 91 implies that the number of non-homeomorphic completely regular topologies on S is the same as the number of equivalence classes of partitions of S, where two partitions $\mathcal{P}$ and $\mathcal{Q}$ are equivalent if there is some permutation $\pi : S \to S$ with $\pi(\mathcal{P}) = \mathcal{Q}$. However, the number of such equivalence classes is *part(n)*, the number of partitions of the integer n. ∎

No such characterization is possible in the infinite case because many different topologies correspond to the same partition. The construction of a partition from a topology given by the $C(s, \tau)$'s works in the infinite case, but this construction will give the same partition for many different topologies. The basic reason for this is that for an infinite set, there are many different topologies that the Tychonoff quotient may have, while in the finite case there is only one – the discrete topology.

Having such a nice relationship between homeomorphic topologies and equivalent partitions, one might ask whether such a relationship is possible between partitions and algebras or between topologies and algebras. The answer is no. The reason is that two finite dimensional algebras are isomorphic if and only if they have the same dimension while two partitions with the same number of equivalence classes do not have to be permutations of each other. Here is an example to illustrate this.

Example Let $S = \{1, 2, 3, 4\}$ and $\mathcal{P} = \{\{1, 2\}, \{3, 4\}\}$ and $\mathcal{Q} = \{\{1, 2, 3\}, \{4\}\}$. Clearly $\mathcal{P}$ and $\mathcal{Q}$ are not permutations of each other, however $\mathcal{A}_P$ is isomorphic to $\mathcal{A}_Q$ since they are both two dimensional. Notice that τ_P is not homeomorphic to τ_Q.

$$\tau_P = \{S, \emptyset, \{1, 2\}, \{3, 4\}\}$$

$$\tau_Q = \{S, \emptyset, \{1, 2, 3\}, \{4\}\}$$

The enumeration of all completely regular topologies on a finite set (our Corollary 93) is also a simple consequence of a formula in [ER]. Let $T^P(n)$ be the number of topologies on a set of cardinality n satisfying property P, let $T_0^P(n)$ the number of such topologies which are also T_0, and let $S(n,m)$ be the number of partitions of a set of cardinality n into m classes (Stirling numbers of the second kind). Then the formula is

$$T^P(n) = \sum_{m=1}^{n} S(n,m) T_0^P(n) \qquad n > 0.$$

(Some restrictions on the property P are necessary, but we need not worry about them here.) If (S,τ) is a finite T_0 completely regular topological space, then (S,τ) is automatically Hausdorff. However, the only Hausdorff topology on a finite set is the discrete topology. Thus, $T_0^P(m) = 1$ for all m, where P is the property of being completely regular, (the property of being completely regular satisfies the required restriction). Thus, $T^P(n) = \sum_{m=1}^{n} S(n,m) = part(n)$.

The reader interested in the enumeration of finite topologies can find more information in the papers [ER, KLRO, KM1, KM2].

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