THE SIZES OF REARRANGEMENTS OF CANTOR SETS

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Abstract. To a linear Cantor set, $C$, with zero Lebesgue measure there is associated the countable collection of the bounded complementary open intervals. A rearrangement of $C$ has the same lengths of its complementary intervals, but with different locations. We study the Hausdorff and packing $h$-measures and dimensional properties of the set of all rearrangements of some given $C$, for general dimension functions $h$. For each set of complementary lengths, we construct a Cantor set rearrangement which has the maximal Hausdorff and the minimal packing $h$-premeasure, up to a constant. We also show that if the packing measure of this Cantor set is positive, then there is a rearrangement which has infinite packing measure.

1. Introduction

Given $E$, a compact subset of the real line contained in the interval $I$, its complement $I \setminus E$ is the union of a countable collection of open intervals, say

$$I \setminus E = \bigcup_j A_j.$$ 

Clearly the intervals $A_j$ determine $E$ but, surprisingly, some geometric information is obtainable from knowing only the lengths (for example, the pre-packing (upper-box) dimension, see [Fal97]) and not the positioning of the $A_j$’s.

In this paper we are interested in singular sets, so we assume that the Lebesgue measure of $E$ is zero. Further, for simplicity, we assume that the endpoints of $I$ are contained in $E$ so that $|I| = |E|$ (where by $|S|$ we mean the diameter of $S \subset \mathbb{R}$). These two assumptions imply that $\sum_n a_n = |I|$, where $a_n = |A_n|$.

For a given positive, summable and non-increasing sequence $a = (a_n)$ there are many possible linear closed sets $E$ such that the complementary intervals have lengths given by the terms of the sequence. Such a rearrangement $E$ will be said to belong to the sequence $(a_n)$ or $E \in \mathcal{C}_a(I)$ (or shortly, $\mathcal{C}_a$). Our main interest lies in the properties of the collection $\mathcal{C}_a$ for a fixed sequence $a$, particularly in the dimensional behaviour as we range over $\mathcal{C}_a$.

These sets were first studied by Borel [Bor49] and Besicovitch and Taylor [BT54]. In their seminal paper, Besicovitch and Taylor studied the $\alpha$-Hausdorff dimension and measures of these cut-out sets. In particular, they proved that

$$\{\dim_H(E) : E \in \mathcal{C}_a\}$$

is a closed interval

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and constructed a Cantor set \( C_a \in \mathcal{C}_a \), as described below, with maximal Hausdorff dimension and measure. Cabrelli et al. [CMMS04] and Garcia et al. [GMS07] continued this study and, among other things, constructed a concave dimension function \( h \) so that \( C_a \) is an \( h \)-set (that is, \( 0 < \mathcal{H}^h(C_a) \leq \mathcal{P}^h(C_a) < \infty \)). Xiong and Wu [XW09] showed that \( \mathcal{C}_a \) is a compact metric space under the Hausdorff distance \( \rho \) and studied density-type properties in \( (\mathcal{C}_a(I), \rho) \). Lapidus and co-workers (see [HL97, LvF06] and the references in these) study these sets under the name “fractal strings” and are especially interested in inverse spectral problems and a surprising relationship with the Riemann zeta function and the Riemann Hypothesis.

We prove a generalization of (1.1) for arbitrary dimension functions \( h \) for both Hausdorff and packing measures. In contrast to the Besicovitch and Taylor result for Hausdorff measure and despite the fact that the (pre)packing dimension of the Cantor set \( C_a \) is maximal over all \( E \in \mathcal{C}_a \), we show that \( C_a \) has the minimal packing \( h \)-premeasure of the sets in \( \mathcal{C}_a \) (up to a constant). Furthermore, if the packing \( h \)-measure of \( C_a \) is positive (such as if \( h(x) = x^s \) when \( C_a \) is an \( s \)-set), then there is some rearrangement \( E \in \mathcal{C}_a \) with infinite packing \( h \)-measure. In fact, \( \{\mathcal{P}^h(E) : E \in \mathcal{C}_a \} \) is either equal to \( \{0\} \) or is equal to \([0, \infty]\). Finally, we also generalize a density result from [XW09] to arbitrary dimension functions.

2. Notation

2.1. The sets \( C_a \) and \( D_a \). There are two sets belonging to a given sequence \( a = (a_n) \) that we will refer to often.

One is built using a Cantor construction and will be denoted by \( C_a \). We begin with a closed interval \( I \) of length \( \sum a_n \) and remove from it an open interval with length \( a_1 \). This leaves two closed intervals, \( I_1^1 \) and \( I_1^2 \), called the intervals of step one. If we have constructed \( \{I_j^k\}_{1 \leq j \leq 2^k} \), the intervals of step \( k \), we remove from each interval \( I_j^k \) an open interval of length \( a_{2^k+j-1} \), obtaining two closed intervals of step \( k+1 \), namely \( I_{j-1}^{k+1} \) and \( I_j^{k+1} \). We define

\[ C_a := \bigcap_{k \geq 1} \bigcup_{1 \leq j \leq 2^k} I_j^k. \]

This process uniquely determines the set \( C_a \). For instance, the position of the first interval to be removed (of length \( a_1 \)) is uniquely determined by the property that the length of the remaining interval on the left is \( a_2 + a_4 + a_5 + a_6 + \cdots \). The classical middle-third Cantor set is the set \( C_a \) associated with the sequence \( a = (a_n) \) where \( a_j = 3^{-n} \) if \( 2^{n-1} \leq j < 2^n \).

The set \( C_a \) is compact, perfect and totally disconnected. The average length of a step \( k \) interval is \( r_k / 2^k \), where \( r_k = \sum_{i \geq n} a_i \). Since the sequence \( (a_n) \) is decreasing, any interval of step \( k-1 \) has length at least the average length at step \( k \), and this, in turn, is at least the length of any interval of step \( k+1 \).

The other important set in the class \( \mathcal{C}_a(I) \) is a countable set that will be denoted by \( D_a \). If \( I = [\alpha, \beta] \), where \( \beta = \alpha + \sum_{j \geq 1} a_j \), and \( x_n = \sum_{j \leq n} a_j \) then

\[ D_a := \{\alpha\} \cup \{\alpha + x_n : n \geq 1\} \cup \{\beta\}. \]

2.2. Dimension functions. We will say that \( h : (0, \infty) \to \mathbb{R} \) is a dimension function if \( h \) is increasing, continuous, doubling, i.e., \( h(2x) \leq ch(x) \), and satisfies \( \lim_{x \to 0} h(x) = 0 \). The class of dimension functions will be denoted \( D \).
Given two dimension functions, $g, h$, we say $g \prec h$ if $\lim_{t \to 0} h(t)/g(t) = 0$ and $g \sim h$ (and say $g$ is comparable to $h$) if there are positive constants $c_1, c_2$ such that $c_1 h(t) \leq g(t) \leq c_2 h(t)$ for $t$ small. We will write $g \preceq h$ if either $g \prec h$ or $g \sim h$.

2.3. **Hausdorff and packing $h$-measures.** For any dimension function $h$, the **Hausdorff** $h$-measure, $\mathcal{H}^h$, can be defined in a similar fashion to the familiar Hausdorff measure (see [Rog98]). Given $E$, a subset of $\mathbb{R}$, we denote by $|E|$ its diameter. A $\delta$-covering of $E$ is a countable family of subsets with diameters at most $\delta$, whose union contains $E$. Define

$$H^h_\delta(E) = \inf \left\{ \sum_{i \geq 1} h(|E_i|) : (E_i) \text{ is a } \delta \text{-covering of } E \right\}$$

and

$$\mathcal{H}^h(E) = \lim_{\delta \to 0} H^h_\delta(E).$$

The $h$-packing measure and premeasure can be defined similarly (see [Tri82]). A $\delta$-packing of a set $E$ is a disjoint family of open intervals, centred at points in $E$, and with diameters at most $\delta$. Define

$$P^h_\delta(E) = \sup \left\{ \sum_{i \geq 1} h(|E_i|) : (E_i) \text{ is a } \delta \text{-packing of } E \right\}.$$

The $h$-packing premeasure, $P^h_0$, is given by

$$P^h_0(E) = \lim_{\delta \to 0} P^h_\delta(E).$$

As $P^h_0$ is not a measure we also define the $h$-packing measure of $E$, $\mathcal{P}^h(E)$, as

$$\mathcal{P}^h(E) = \inf \left\{ \sum_i P^h_0(E_i) : E = \bigcup_{i=1}^{\infty} E_i \right\}.$$ 

Clearly, $\mathcal{P}^h(E) \leq P^h_0(E)$ for any set $E$ and since $h$ is doubling, $\mathcal{H}^h(E) \leq \mathcal{P}^h(E)$ ([TT85]).

In the special case when $h_s(x) = x^s$, then $\mathcal{H}^{h_s}$ is the usual $s$-dimensional Hausdorff measure and similarly for $s$-packing (pre)measure.

For a given set $E$ put

$$N(E, \varepsilon) = \min\{k : E \subset \bigcup_{i=1}^{k} B(x_i, \varepsilon)\} \text{ and } P(E, \varepsilon) = \max\{k : \exists \text{ disjoint } (B(x_i, \varepsilon))_{i=1}^{k} \text{ with } x_i \in E\}.$$

Elementary geometric reasoning shows that for any set $E$

$$N(E, 2\varepsilon) \leq P(E, \varepsilon) \leq N(E, \varepsilon/2). \quad (2.1)$$

Furthermore, it is obvious that

$$\mathcal{H}^h(E) \leq \liminf_r N(E, r) h(r) \text{ and } \mathcal{P}^h_0(E) \geq \limsup_r P(E, r) h(r).$$

Also, if $f \preceq h$, then for any set $E$ there is a constant $c$ such that $\mathcal{H}^h(E) \leq c \mathcal{H}^f(E)$, and similarly for packing (pre)measures.
The upper box dimension of $E$ is given by

$$\limsup_{r \to 0} \frac{\log(N(E, r))}{-\log r} = \limsup_{r \to 0} \frac{\log(P(E, r))}{-\log r}$$

and is known to coincide with the pre-packing dimension of $E$, i.e., the index given by the formula $\inf \{s : P^s_0(E) = 0\}$ [Tri82].

3. Hausdorff measures of rearrangements

In [BT54], Besicovitch and Taylor gave bounds for the Hausdorff $s$-measures of Cantor sets $C_a$ in terms of the asymptotic rate of decay of the tail sums,

$$r_n = \sum_{i \geq n} a_i,$$

of the sequence. In [GMS07], those estimates were extended to $h$-Hausdorff and packing premeasures.

**Theorem 3.1** (GMS07). Suppose $h \in \mathcal{D}$. Then

1. $1/4 \liminf_{n \to \infty} nh(r_n/n) \leq \mathcal{H}^h(C_a) \leq 4 \liminf_{n \to \infty} nh(r_n/n)$
2. $1/8 \limsup_{n \to \infty} nh(r_n/n) \leq \mathcal{P}^h(C_a) \leq 8 \limsup_{n \to \infty} nh(r_n/n)$.

A set $E$ is called an $s$-set if $0 < \mathcal{H}^s(E) \leq \mathcal{P}^s(E) < \infty$. Although not all Cantor sets $C_a$ are $s$-sets, Cabrelli et al. [CMMS04] proved that for any non-increasing sequence $(a_n)$ there is a concave function $h_a \in \mathcal{D}$ such that $h_a(r_n/n) \sim 1/n$. Thus $C_a$ is an $h_a$-set. Any function with the property $h(r_n/n) \sim 1/n$ is called an associated dimension function and all associated dimension functions for a given sequence $a$ are comparable. The set $C_a$ has Hausdorff and packing $h$-premeasure finite and positive if and only if $h$ is an associated dimension function [CHM10].

Given $E \subseteq \mathbb{R}$ and $\varepsilon > 0$, let

$$E(\varepsilon) = \{x \in \mathbb{R} : |x - y| < \varepsilon \text{ for some } y \in E\}.$$

Falconer [Fal97, 3.17] observed that if $E, E' \in \mathcal{C}_a$ then $\mathcal{L}(E(\varepsilon)) = \mathcal{L}(E'(\varepsilon))$ where $\mathcal{L}$ denotes the Lebesgue measure. Observe that any union of $\varepsilon$-balls, with centres in $E$, is contained in $E(\varepsilon)$ and any union of $2\varepsilon$-balls covers $E(\varepsilon)$ if the union of the $\varepsilon$-balls with the same centres covers $E$. Thus we have

$$P(E, r)2r \leq \mathcal{L}(E(r)) \leq N(E, r)4r. \quad (3.1)$$

Combining (2.1) and (3.1) gives the following useful geometric fact.

**Lemma 3.2.** For any $E \in \mathcal{C}_a$ and $\varepsilon > 0$,

$$P(C_a, \varepsilon) \leq 2N(E, \varepsilon) \leq 2P(E, \varepsilon/2) \leq 4N(C_a, \varepsilon/2).$$

In [BT54] they show that $C_a$ has maximal $\mathcal{H}^s$ measure in $\mathcal{C}_a$. Our first result extends this (up to a constant) for arbitrary $h$. We remark that if $h$ is assumed to be concave, the same arguments as given in [BT54] show that $\mathcal{H}^h(E) \leq \liminf_{n \to \infty} nh(r_n/n)$ for any $E \in \mathcal{C}_a$.

**Proposition 3.3.** If $h \in \mathcal{D}$ and $E \in \mathcal{C}_a$, then $\mathcal{H}^h(E) \leq c\mathcal{H}^h(C_a)$ where $c$ depends only on the doubling constant of $h$. 
Proof. Since $h$ is a doubling function, the lemma above together with the definitions of $\mathcal{H}^h$ and $N(E, r)$ implies

$$\mathcal{H}^h(E) \leq \liminf_{r \to 0} N(E, r)h(r) \leq c \liminf_{r \to 0} N(C_a, r)h(r).$$

Temporarily fix $r > 0$ and choose $n$ such that

$$\frac{r^{2^{n+1}}}{2^{n-1}} \geq r \geq \frac{r^{2^n}}{2^n}.$$

Since the length of any Cantor interval at step $n + 1$ is at most the average of the lengths of the step $n$ intervals, the $2^n + 1$ intervals centred at the right end points of the Cantor intervals of step $n + 1$ and radii $r^{2^n}/2^n$ cover $C_a$. Thus $N(C_a, r) \leq 2^{n+1} + 1$ and hence

$$N(C_a, r)h(r) \leq 2^{n+1}h\left(\frac{r^{2^{n+1}}}{2^{n-1}}\right) \leq 4 \cdot 2^{n-1}h\left(\frac{r^{2^n}}{2^n}\right).$$

Therefore Theorem 3.1 implies

$$\mathcal{H}^h(C_a) \geq \frac{1}{4} \liminf_{n \to \infty} 2^n h\left(\frac{r^{2^n}}{2^n}\right) \geq \frac{1}{16} \liminf_{r \to 0} N(C_a, r)h(r) \geq \frac{1}{16c} \mathcal{H}^h(E).$$

Remark 3.4. If $C_a$ corresponds to a middle-$\tau$ Cantor set, then $\mathcal{H}^s(C_a) = 1 = \liminf_n n(r_n/n)^s$ where $s = -\log 2/\log(\tau)$. Thus the comment immediately before the proposition shows we may take $c = 1$ in the proposition and $C_a$ has the maximal $\mathcal{H}^s$ measure amongst $E \in \mathcal{G}_a$ in this case. For the general case, it is unknown what the minimal constant $c$ is and which set $E \in \mathcal{G}_a$ (if any) has the maximum Hausdorff measure.

Besicovitch and Taylor, [BT54], also show that if $s < \dim_H C_a$, then for any $\gamma \geq 0$ there is a rearrangement $E$ such that $\mathcal{H}^s(E) = \gamma$. We extend this result to dimension functions and also prove that, in addition, $E$ can be chosen to be perfect.

Theorem 3.5. Let $I$ be an interval with $|I| = \sum a_i$. If $h \prec h_0$ and $\gamma \geq 0$, then there is a perfect set $E \in \mathcal{G}_a(I)$ such that $\mathcal{H}^h(E) = \gamma$.

Proof. As shown in [CHM10], the assumption $h \prec h_0$ implies that $\mathcal{H}^h(C_a) = \infty$, thus by [Lar67] there exists a closed subset $E \subset C_a$ with $\mathcal{H}^h(E) = \gamma$. The set $E$ might not be perfect or belong to the sequence $(a_n)$, so we will modify it in order to obtain the desired properties.

Since both $E$ and $C_a$ are closed, there are collections of open intervals $A_j$ and $(\alpha_j, \beta_j)$ such that

$$I \setminus C_a = \bigcup_{i \geq 1} A_i \quad I \setminus E = \bigcup_{j \geq 1} (\alpha_j, \beta_j).$$

Fix $j \geq 1$ and define $\Lambda_j = \{i : (\alpha_j, \beta_j) \supset A_i\}$. Of course, $\sum_{i \in \Lambda_j} |A_i| = \sum_{i \in \Lambda_j} a_i = \beta_j - \alpha_j$. Since $C_a$ is perfect, $\Lambda_j$ is either a singleton or infinite. In the first case the length of the gap $(\alpha_j, \beta_j)$ is a term of the sequence $(a_n)$.

If, instead, $\Lambda_j$ is infinite, consider the terms $\{a_i : i \in \Lambda_j\}$ in decreasing order and call this subsequence $a^{(j)}$. For each fixed $j$, we will decompose the subsequence $a^{(j)}$ into countably many subsubsequences $a^{(j,k)}$ for $k = 1, 2, \ldots$.

First, fix a sequence $d_n$ such that $h(d_n) \leq n^{-2}$. We start by defining $a^{(j,1)}$ and begin by putting $a^{(j,1)}_1 = a^{(j)}_1$. Assume $a^{(j,1)}_i$ are defined for $i = 1, 2, \ldots, m - 1$ and $a^{(j,1)}_m = a^{(j)}_N$. Pick the first integer $N > N'$ satisfying $a^{(j)}_N \leq d_m - d_{m+1}$ and define...
\[ a^{(j,k)}_{m} = a^{(j)}_{N+m}. \]  
(We do not just take \( a^{(j)}_N \) in order to have enough terms to build \( a^{(j,k)} \) for \( k \geq 1 \).)

Now inductively assume \( a^{(j,k)} \) have been defined for \( k = 1, 2, \ldots, m - 1 \). We let \( a^{(j,m)}_{1} \) be the first term of \( a^{(j)} \) that was not picked in \( a^{(j,k)} \) for \( k < m \). If also the terms \( a^{(j,m)}_i \) are defined for \( i = 1, 2, \ldots, l - 1 \), pick \( N \) to be the first integer satisfying:

(i) \( a^{(j)}_N \) is not an element of one of the sequences \( a^{(j,k)} \), \( k = 1, \ldots, m - 1 \), that have already been defined;

(ii) \( N > N' \) where \( N' \) is defined by \( a^{(j,m)}_{l-1} = a^{(j)}_{N'} \); and

(iii) \( a^{(j)}_{N'} \leq d_l - d_{l+1} \).

Then put \( a^{(j,m)}_k = a^{(j)}_k \) where \( k \geq N + l \) is the minimal index not already chosen.

Note that the union of \( a^{(j,k)} \) for \( k = 1, 2, \ldots \) is \( a^{(j)} \) and by (iii),

\[
\liminf_{n \to \infty} n h \left( \sum_{i \geq n} a^{(j,k)}_i/n \right) = 0 \quad \text{for all } j, k.
\]

Inside each interval \([ \alpha_j, \beta_j ]\) consider the subintervals \( I^{(j)}_m \) with length equal to \( \sum a^{(j,m)}_i \). Construct within each such subinterval the Cantor set \( C^{(j,m)} \) associated to the sequence \( a^{(j,m)} \). By Theorem 3.1 we have \( h^h(C^{(j,k)}) = 0 \) for any pair \((j,k)\). The set \( E \cup (\cup_{j,m} C^{(j,m)}) \) is perfect, belongs to \( C_a(I) \) and has the same Hausdorff \( h \)-measure as \( E \). □

A direct consequence of this Theorem is the following extension of Theorem 2 in [XW09]. Let \( \rho \) be the Hausdorff metric defined for compact subsets of the real line by

\[
\rho(A, B) = \max \{ \sup_{y \in B} \inf_{x \in A} d(x, y), \sup_{x \in A} \inf_{y \in B} d(x, y) \}.
\]

In [XW09] it is shown that \( (C_a(I), \rho) \) is compact.

**Corollary 3.6.** Let \((a_n)\) be a decreasing, positive and summable sequence and let \( I \) be an interval with \(|I| = \sum a_k\). If \( h < h_a \) and \( \gamma \geq 0 \), the set \( \Gamma = \{ E \in C_a(I) : h^h(E) = \gamma \} \) is dense in \( C_a(I) \) with the Hausdorff metric.

**Proof.** Fix \( E \in C_a(I) \) and \( n \in \mathbb{N} \). As usual, assume \( I \setminus E = \bigcup A_j \) where the lengths of \( A_j \) are decreasing. There is a permutation \( \sigma \) of \( \{1, \ldots, n\} \) (determined by \( n \) and \( E \)) such that \( A_{\sigma(1)}, \ldots, A_{\sigma(n)} \) are placed from left to right, meaning that if \( x_i \in A_{\sigma(i)} \) then \( x_1 < x_2 < \cdots < x_n \). We define a subfamily of \( C_a(I) \) by

\[
\mathcal{C}_a^n(I) = \{ F \in C_a(I) : x_1 \in A^F_{\sigma(i)}, \text{ then } x_1 < x_2 < \cdots < x_n \}
\]

where \( \{A^F_i\} \) are the intervals (in order of decreasing lengths) whose union is the complement of \( F \).

In [XW09] the authors proved that \( \text{diam}(\mathcal{C}_a^n(I)) \leq 3r_{n+1} \), thus it is enough to prove that \( \Gamma \cap \mathcal{C}_a^n(I) \neq \emptyset \).

Put \( I = [\alpha, \beta], \bar{I} = [\alpha + \sum_{k=0}^n a_k, \beta] \) and \( \bar{a}_k = a_{n+k} \). By Theorem 3.5, there is a set \( \bar{E} \in C_a(I) \) with \( h^h(\bar{E}) = \gamma \). The set \( F = \{ \alpha + \sum_{k=0}^j a_k : 0 \leq j \leq n \} \cup \bar{E} \) belongs to \( \Gamma \cap \mathcal{C}_a^n(I) \). □
4. Packing measures and packing premeasures of rearrangements

In contrast to the case for Hausdorff measure, it was shown in [Fal97] that the pre-packing dimension is the same for any set \( E \in \mathcal{E}_a \). Furthermore, as we show next, the packing premeasure of the Cantor set is (up to a constant) the least premeasure of any set with the same gap lengths. This result is dual to Proposition 3.3.

**Proposition 4.1.** There is a constant \( c \) such that if \( E \in \mathcal{E}_a \), then \( \mathcal{P}^h_0(C_a) \leq c \mathcal{P}^h_0(E) \).

**Proof.** Similar arguments to Prop. 3.3 show that

\[
\mathcal{P}^h_0(C_a) \leq c \limsup_{r \to 0} P(C_a, r) h(r).
\]

But \( P(C_a, r) \leq 2P(E, r/2) \) for any \( E \in \mathcal{E}_a \) and for any set \( E \), \( \limsup_{r \to 0} P(E, r) h(r) \) is a lower bound for \( \mathcal{P}^h_0(E) \). Combine these observations. \( \square \)

As is the case with Hausdorff measures, the sharp value of \( c \) and the exact set from \( \mathcal{E}_a \) which minimizes \( \mathcal{P}^h_0 \) is unknown, even in the case of the middle-1/3 Cantor set \( C_a \). It is known that \( 4^s = \mathcal{P}^h(C_a) \neq \limsup_n n(r_n/n)^s = 1 \), where \( s = \log 2/\log 3 \) (see [FHWW00, Fen03]).

**Corollary 4.2.** If \( \mathcal{P}^h_0(C_a) > 0 \) then \( \mathcal{P}^h_0(D_a) = \infty \). In particular, \( \mathcal{P}^h_0(D_a) = \infty \).

**Proof.** Since \( D_a \) is countable, \( \mathcal{P}^h(D_a) = 0 \) (for any \( h \)). By virtue of the previous proposition, for this particular \( h \) we have \( \mathcal{P}^h_0(D_a) > 0 \). It was proven in [WW04] that if \( \mathcal{P}^h_0(D_a) < \infty \), then \( \mathcal{P}^h_0(D_a) \leq c \mathcal{P}^h(D_a) \) for a suitable constant \( c \). But this is not the case. \( \square \)

From here on we will be more restrictive with the dimension functions and require, in addition, that they are subadditive, i.e., there is a constant \( C \) such that \( h(x + y) \leq C(h(x) + h(y)) \) for all \( x, y \). In [Pee70] Peetre showed that any function equivalent to a concave function is subadditive. Since every sequence admits an associated dimension function that is concave [CMMS04], any function \( h \) which makes \( C_a \) an \( h \)-set will be subadditive.

**Lemma 4.3.** If \( h \in \mathcal{D} \) is subadditive, then \( \mathcal{P}^h_0(E) \leq 2 \mathcal{P}^h_0(D_a) \) for all \( E \in \mathcal{E}_a \).

**Proof.** Without loss of generality \( 0 \in I \) and \( E = I \setminus \cup_{j \geq 1} A_j \) where \( A_j \) are open intervals with decreasing lengths, \( |A_j| = a_j \). Consider any \( \delta \)-packing of \( E \), say \( \{B_j\} \). For each \( j \), let \( \Delta_j = \{i : A_i \cap B_j \text{ is not empty}\} \).

Let \( B'_j \) denote the interval centered at \( x_i = \sum_{n \leq i} a_n \) (where \( x_0 = 0 \)) and diameter equal to \( \min(a_{i+1}, \delta) \). The balls, \( \{B'_j\}, i = 0, 1, 2, \ldots \) form a \( \delta \)-packing of \( D_a \), thus \( \sum h(|B'_j|) \leq \mathcal{P}^h_0(D_a) \).

By subadditivity,

\[
\sum h(|B_j|) \leq \sum_{j \in \Delta_j} \sum_{i \in \Delta_j} h(|A_i \cap B_j|) \leq \sum_{j \in \Delta_j} h(\min(a_i, \delta)) \leq 2 \sum_{i \geq 1} h(|B'_{i-1}|) \leq 2 \mathcal{P}^h_0(D_a),
\]

where the penultimate inequality holds because each \( i \) belongs to \( \Delta_j \) for at most two choices of \( j \). Since \( \{B_j\} \) was an arbitrary \( \delta \)-packing of \( E \), the result follows. \( \square \)
It is known that for Cantor sets, \( C_a \), the packing dimension coincides with the pre-packing dimension ([CHM10]). Since the pre-packing dimension of all sets in \( C_a \) coincide and the pre-packing dimension is an upper bound for the packing dimension of a set, it follows that \( \dim_p C_a \geq \dim_p E \) for any \( E \in C_a \). Despite this, we have the following theorem.

**Theorem 4.4.** If \( h \in \mathcal{D} \) is subadditive the following statements are equivalent:

1. There exists a set \( E \in C_a \) with \( P^h_0(E) > 0 \);
2. \( P^h_0(D_a) = \infty \);
3. \( \sum h(a_i) = \infty \);
4. There exists a perfect set \( E \in C_a \) with \( \mathcal{P}^h(E) = \infty \).

**Proof.**

4 \( \Rightarrow \) 1 is trivial as \( P^h_0(E) \geq P^h(E) \).

1 \( \Rightarrow \) 2 By Lemma 4.3, \( P^h_0(D_a) > 0 \) and this forces \( P^h_0(D_a) = \infty \) as in Corollary 4.2.

2 \( \Rightarrow \) 3 Since \( P^h_0(D_a) = \infty \), given \( \delta > 0 \) and \( M \) there is a \( \delta \)-packing of \( D_a \), say \( \{B_i\} \), such that \( \sum h(|B_i|) \geq M \). Put \( \Delta_j = \{i : (x_i, x_{i+1}) \cap B_j \neq \emptyset\} \). Since a gap of \( D_a \) can intersect at most two of these intervals \( B_j \), we have

\[
\sum_j h(|B_j|) \leq \sum_j \left( \sum_{i \in \Delta_j} (x_i, x_{i+1}) \right) \leq \sum_j \sum_{i \in \Delta_j} h(x_i, x_{i+1}) \leq 2 \sum_i h(a_i)
\]

and therefore the series \( \sum h(a_i) \) is divergent.

3 \( \Rightarrow \) 4 Take the interval \( I_0 = [0, \sum a_i] \). Choose \( N_0 \) such that \( \sum_{1 \leq i \leq N_0-1} h(a_i) \geq 1 \) and remove from \( I_0 \) a total of \( N_0 - 1 \) open intervals with lengths \( a_1, \ldots, a_{N_0-1} \), respectively, where we remove these intervals in order from left to right. This produces \( N_0 \) closed intervals, denoted by \( I^1_j \) for \( j = 1, \ldots, N_0 \), which we will call the intervals of step one.

Put \( N^1_0 = N_0 \) and for \( 1 \leq j \leq N_0 \), choose \( N^1_j \) such that

\[
\sum_{N^1_{j-1} \leq i \leq N^1_j} h(a_i) \geq 2.
\]

From each \( I^1_j \) we remove \( N^1_j - N^1_{j-1} - 1 \) open intervals with lengths \( a_i \) for \( i = N^1_{j-1}, \ldots, N^1_j - 1 \), again removing them in order from left to right. This produces a total of \( S_1 := N^1_{N_0} - N_0 \) closed intervals of step 2 that will be labeled \( (I^2_j)_{1 \leq j \leq S_1} \).

We proceed inductively and assume we have constructed \( S_{k-1} \) intervals of step \( k \), \( I^k_1, \ldots, I^k_{S_{k-1}} \), put \( N^k_0 = N^{k-1}_{S_{k-1}} \) and for \( j = 1, \ldots, S_{k-1} \) pick \( N^k_j \) such that

\[
\sum_{i = N^k_{j-1}}^{N^k_j - 1} h(a_i) \geq 2^k.
\]

From \( I^k_j \) remove, from left to right, \( N^k_j - N^k_{j-1} - 1 \) intervals of lengths \( a_i \) for \( i = N^k_{j-1}, \ldots, N^k_j - 1 \) obtaining \( S_k := N^k_{S_k} - N^k_0 \) closed intervals of step \( k + 1 \), denoted \( (I^{k+1}_j)_{1 \leq j \leq S_k} \).

Put

\[
E = \bigcap_{k \geq 1} \bigcup_{1 \leq j \leq S_k} I^{k+1}_j \in C_a.
\]

As with the construction of \( C_a \), the fact that \( |I| = \sum a_j \) ensures that this construction uniquely determines \( E \). Clearly, \( E \in C_a \) and is perfect.
We claim that $P^h(E) = \infty$. To see this, suppose that $E \subset \bigcup E_i$ with $E_i$ closed. By Baire’s Theorem there is (at least) one $E_i$ with non-empty interior and therefore one of the sets $E_i$ contains an interval from some step in the construction. It follows that in order to prove $P^h(E) = \infty$, it is enough to prove that $P^h_0(E \cap I_j^k) = \infty$ for any interval $I_j^k$.

Fix such an interval $I_j^k$. It will be enough to show that for any $\delta > 0$ and $M$ there is a $\delta$-packing, $\{B_i\}$, of $E \cap I_j^k$ with $\sum h(|B_i|) \geq M$. Pick $K$ such that $a_j < \delta$ if $j \geq K$, $2^K \geq M$ and $K \geq k$. Inside $I_j^k$ take an interval of step $K$, say $I_j^K$. Denote by $A_i = (\alpha_i, \beta_i)$ the gap with length $a_i$. For $i = N_j^K - 1, \ldots, N_j^K - 1$ the gaps $A_i$ are inside the interval $I_j^K$. Now take the $\delta$-packing $B_i = (\alpha_i - a_i/2, \alpha_i + a_i/2)$ for $N_j^K - i \leq N_j^K$. These sets satisfy

$$\sum_{i=N_j^K-1}^{N_j^K-1} h(|B_i|) \geq \sum_{i=N_j^K-1}^{N_j^K-1} h(a_i) \geq 2^K \geq M.$$

Since the associated dimension function $h_\alpha$ is subadditive and $P^h_0(C_\alpha) > 0$, we immediately obtain the following corollary.

**Corollary 4.5.** There exists $E \in \mathcal{C}_\alpha$ such that $P^h_\alpha(E) = \infty$.

For example, if $C_\alpha$ is the classical middle third Cantor set, then there exists $E \in \mathcal{C}_\alpha$ such that $P^h_\alpha(E) = \infty$ for $s = \log 2/\log 3$.

One can even find functions $f > h_\alpha$ for which this is true.

**Example 4.6.** Take $\{a_n\} = \{n^{-1/p}\}$ for $p < 1$; the associated dimension function is $h_\alpha(x) = x^p$. If we put $f(x) = x^p/|\log x|$, then $f/h \to 0$ as $x \to 0$, $f$ is concave and $\sum f(a_n) = \infty$. Hence $P^h_0(D_\alpha) = \infty$ and $P^h(E) = \infty$ for some $E \in \mathcal{C}_\alpha$.

However, since all sets with the same gap lengths have the same pre-packing dimension there is a severe restriction on the functions $f$ with the property above.

**Proposition 4.7.** Suppose $P^h_0(E) > 0$ for some $E \in \mathcal{C}_\alpha$. Then $\liminf \frac{\log f}{\log h_\alpha} \leq 1$.

**Proof.** Our proof is a modification of Lemma 3.7 in [Fal90].

If the conclusion is not true, then for some $s > 1$ and suitably small $x$ we have $f(x) \leq h_\alpha^s(x)$.

Assume $P^h_0(E) \geq \varepsilon > 0$. For each $\delta > 0$ there are disjoint balls $\{B_i\}$, with diameter at most $\delta$ and centred in $E$, such that $\sum f(|B_i|) \geq \varepsilon$. For each $k$, let $n_k$ denote the number of balls $B_i$ with $r_{2k+1}/2^{k+1} \leq |B_i| < r_{2k}/2^k$. In terms of this notation we have

$$\varepsilon \leq \sum_i f(|B_i|) \leq \sum_i h_\alpha^s(|B_i|) \leq \sum_k n_k h_\alpha^s(r_{2k}/2^k) \leq \sum_k n_k 2^{-ks}$$

and $P(E, r_{2k+1}/2^{k+1}) \geq n_k$. 


Fix $t \in (1, s)$. The previous inequality implies that $n_k \geq \varepsilon 2^{kt}(1 - 2^{-s})$ for infinitely many $k$. For such $k$,
\[
\limsup_k \varepsilon 2^{kt}(1 - 2^{-s})2^{-(k+1)} \leq \limsup_k n_k 2^{-(k+1)}
\]
\[
\leq \limsup_k P(E, \frac{r_{2k+1}}{2^{k+1}}) h_a(\frac{r_{2k+1}}{2^{k+1}})
\]
\[
\leq \mathcal{P}_0^h(C_a) < \infty.
\]
But since $t > 1$, the left hand side of this inequality is $\infty$ and this is a contradiction.

We finish with analogues of Theorem 3.5 and Corollary 3.6 for packing measure.

**Theorem 4.8.** Suppose $h \preceq h_a$ and $\gamma > 0$. There is a perfect set $E \in \mathcal{C}_a$ with $\mathcal{P}^h(E) = \gamma$.

**Proof.** Corollary 4.5 implies that there is a perfect set $E \in \mathcal{C}_a$ with $\mathcal{P}^h(E) = \infty$. Analogous reasoning to that used in the proof of Theorem 3.5 shows that it will be enough to establish that for any fixed $\gamma$ there is a closed subset of $E$ of $h$-packing measure $\gamma$. In [JP95], Joyce and Preiss prove that if a set has infinite $h$-packing measure (for any $h \in D$) then the set contains a compact subset with finite $h$-packing measure. With a simple modification of their proof, in particular Lemma 6, we obtain a set of finite packing measure greater than $\gamma$. Then using standard properties of regular, continuous measures we get the desired closed set.

**Corollary 4.9.** Let $a$ be a decreasing, positive and summable sequence and let $I$ be an interval with $|I| = \sum a_k$. If $h \preceq h_a$, the set $\{E \in \mathcal{C}_a(I) : \mathcal{P}^h(E) = \gamma\}$ is dense in $(\mathcal{C}_a(I), \rho)$.

**Proof.** The proof is analogous to the one of Corollary 3.6.

**References**


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