Differential Equations using Generalized Derivatives on Fractals

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Abstract In a previous paper [11] we introduced the notion of a μ-derivative and showed how to formulate differential equations in terms of this derivative. In this paper, we extend this approach to the definition of a weak derivative which provides a framework for solving variational problems with respect to fractal measures. We apply our method to a specific boundary value problem, namely a 1D eigenvalue problem over a fractal measure.

1 Introduction: Derivatives with respect to a fractal measure

In this paper we present a framework for solving variational problems with respect to a fractal measure by extending the ideas from [11]. Our theory uses the weak formulation and thus we define the weak derivative and the resulting Sobolev spaces in the natural way. For the one-dimensional problems we discuss in this paper, the variational problems can be transformed by an appropriate change-of-variable into a problem involving Lebesgue measure and thus many of the classical results can be used directly. Problems in higher dimensions require a substantial reworking of the classical theory and are the subject of a future paper in preparation.

In a previous paper [11] we introduced the notion of a μ-derivative and we discussed how to formulate differential equations in which the derivative is replaced

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by a μ-derivative. We considered the equivalent integral equation,

$$u(x) = u_0 + \int_0^x f(t, u(t))\, d\mu(t)\,, \tag{1}$$

where μ is a fractal (Borel probability) measure, assumed to be nonatomic, on $[0,1]$. We studied the existence and uniqueness of solutions to these fractal integral equations based on the Picard operator. Our main interest was in the fractal nature of the solutions and we used Iterated Function Systems (IFS) to investigate the behaviour and self-similarity of these solutions. As usual we can try to formulate an integral equation into an equivalent differential form. Motivated by this we defined the μ*-derivatives* of a function G to be

$$D_\mu^+(G)(x) := \lim_{h\to 0^+} \frac{G(x+h)-G(x)}{\mu([x,x+h])}\,.$$

In a similar way, we can define

$$D_\mu^-(G)(x) := \lim_{h\to 0^+} \frac{G(x)-G(x-h)}{\mu([x-h,x])}\,.$$

Whenever the two limits are equal we label their common value $D_\mu(G)(x)$ and say that G *is* μ*-differentiable at* x [11].

A version of the Fundamental Theorem of Calculus holds [11] so that the integral equation (1) becomes the μ-differential initial value problem,

$$D_\mu(u)(x) = f(x, u(x)), \quad u(0) = u_0. \tag{2}$$

The following results are very useful when dealing with calculations involving the notion of μ-derivative.

Proposition 1 ([11]). *Let us suppose that* μ *is non-atomic and let* $F : K = \mathrm{supp}(\mu) \to [0,1]$ *be the cumulative of* μ *and* $F^{-1} : [0,1] \to K$ *be its inverse. Then, given a function* $f : K \to \mathbb{R}$*, the following change of variable rule holds:*

$$\int_K f(x)d\mu(x) = \int_0^1 f(F^{-1}(x))dx\,, \tag{3}$$

where dx indicates integration over Lebesgue measure on [0,1].

Proposition 2 ([11]). *Let us suppose that* μ *is non-atomic and let* $F : K = \mathrm{supp}(\mu) \to [0,1]$ *be the cumulative of* μ *and* $F^{-1} : [0,1] \to K$ *be its inverse. Then, given a function* $f : K \to \mathbb{R}$*, the following chain rule holds*

$$D_\mu f(y) = \frac{d}{dx} f(F^{-1}(x))|_{x=F(y)},\ \textit{for } \mu\textit{-a.e. } y, \tag{4}$$

where dx denotes Lebesgue measure and $y = F^{-1}(x)$*. Moreover, the following formula for higher-order derivatives holds:*

$$D_\mu^n f(y) = \frac{d^n}{dx^n} f(F^{-1}(x))|_{x=F(y)}. \quad (5)$$

Using these properties it is not hard to show that the following version of integration by parts holds (where $\mathrm{cov}(A)$ is the convex hull of A).

Proposition 3. *Let us suppose* $[a,b] = \mathrm{cov}(\mathrm{supp}(\mu))$. *Then the following formula holds:*

$$\int_a^b D_\mu f(t)\, g(t)\, d\mu(t) = f(b)g(b) - f(a)g(a) - \int_a^b f(t) D_\mu g(t)\, d\mu(t). \quad (6)$$

In the next sections we extend this approach to deal with boundary value problems (BVP) and with particular application to a simple example. We then introduce the notion of a weak μ-derivative and present a variational formulation of the BVP.

The paper is organized as follows. Section 2 presents the notion of μ-weak derivative and the definition of the Hilbert space $H^1_\mu(K)$ along with an application to a one-dimensional eigenvalue BVP. Section 3 recalls the basic definitions of Iterated Function Systems and the notion of attractor. Section 4 presents some convergence results and Section 5 contains some concluding remarks.

We provide a brief excursion into this topic with the intention to interest the reader in the possibilities. Because of space limitations we do not provide proofs. For a much more in-depth discussion, including proofs and extensions we invite the reader to read our forthcoming paper (in preparation).

It is important to mention that our work here (as in [11]) is strongly related to other previous work in analysis on *time-scales* (see [8, 2] and the references therein), in *measure differential equations* (see [17, 3] and the references therein) and also in Stieltjes derivatives (as nicely explained in [16]). More recent work in time-scale analysis which is strongly related to the current paper can be found in [4] (and its references). The papers [13, 14] present another method for defining calculus on subsets of $E \subset \mathbb{R}$ which is geometrically defined and intrinsic to E (and so do not depend on the existence of a measure on E). Their results imply the results in [11] in the case of a "uniform" measure on E and thus could be used as an alternative approach to ours.

From the perspective of applications, the use of fractal derivatives in physics has been recognized, for example, in [6] as have been variational methods [7]. There is an enormous literature on the subject which this paper cannot hope to address even in part. Here we simply mention [18], [10] and [5] as noteworthy contributions to the field.

2 The weak formulation and $H^1_\mu(K)$

Let $K \subset \mathbb{R}$ be a given compact "fractal" set with convex hull $\mathrm{cov}(K) = [a,b]$ and μ be a Borel probability measure supported on it. For a given function $\phi : K \to \mathbb{R}$, we denote by $D_\mu \phi$ the μ-derivative of ϕ with respect to μ at x (which is well-defined at μ-almost all x). In the sequel we denote by $C_c^1(K)$ the set of all functions for which the μ-derivative exists, it is continuous and $\phi(a)=\phi(b)=0$. Given a function $u: K \to \mathbb{R}$, the *weak μ-derivative* of u is a function $g: K \to \mathbb{R}$ which satisfies

$$\int_K u D_\mu \phi \, d\mu = -\int_K g\phi \, d\mu \,, \quad \text{for all } \phi \in C_c^1(K) \,. \tag{7}$$

We also denote the weak μ-derivative of u by $D_\mu u$. Using the fact the μ is supported on K, the previous integral can be rewritten as

$$\int_{[a,b]} u D_\mu \phi \, d\mu = -\int_{[a,b]} g\phi \, d\mu \,, \tag{8}$$

where μ also denotes its extension to $[a,b]$ (i.e., the measure $\mu(A) = \mu(K \cap A)$).

As usual, we define the space $L_\mu^p(K)$ as the set of all functions $u: K \to \mathbb{R}$ that satisfy the condition,

$$\int_K |u|^p d\mu < +\infty. \tag{9}$$

In a similar way, we denote by $W_\mu^{1,p}(K)$ the following set,

$$W_\mu^{1,p}(K) = \left\{ u: K \to \mathbb{R}, u \in L_\mu^p(K) : D_\mu u \text{ exists and } D_\mu u \in L_\mu^p(K) \right\} \,, \tag{10}$$

with $H_\mu^1(K) = W_\mu^{1,2}(K)$. It is not complicated to prove that the space $H_\mu^1(K)$ is Hilbert with respect to the inner product,

$$< u, v >= \int_K D_\mu u(x) D_\mu v(x) d\mu + \int_K u(x) v(x) d\mu \,,$$

and induced norm

$$\|u - v\|_{H_\mu^1} = \|D_\mu u - D_\mu v\|_{L_\mu^2(K)} + \|u - v\|_{L_\mu^2(K)}.$$

Example: We now consider the Dirichlet problem taking the form,

$$D_\mu^2 u(x) + \lambda u(x) = f(x), \quad u(0) = 0, \; u(1) = 0. \tag{11}$$

Following the standard procedure, we obtain an equivalent formulation by first multiplying both sides by a test function $\xi \in C_c^1(K)$. Integration by parts leads to

$$\begin{aligned}
\int_K f(x)\xi(x) d\mu(x) \quad &= \quad \int_K D_\mu^2 u(x)\xi(x) d\mu(x) + \lambda \int_K u(x)\xi(x) d\mu(x) \\
&= \quad D_\mu u(b)\xi(b) - D_\mu u(a)\xi(a) + \int_K D_\mu u(x) D_\mu \xi(x) d\mu(x) \\
&\quad + \lambda \int_K u(x)\xi(x) d\mu(x) \\
&= \quad \int_K D_\mu u(x) D_\mu \xi(x) d\mu(x) + \lambda \int_K u(x)\xi(x) d\mu(x) \,.
\end{aligned}$$

We arrive at the variational form,

$$\int_K D_\mu u D_\mu \xi \, d\mu + \lambda \int_K u\xi \, d\mu = \int_K f\xi \, d\mu. \tag{12}$$

If we define the bilinear form,

$$b(u,v) := \int_K D_\mu u(x) D_\mu v(x) d\mu(x) + \lambda \int_K u(x)v(x)d\mu(x)\,, \tag{13}$$

and the linear form,

$$\theta(v) = \int_K f(x)v(x)d\mu(x)\,, \tag{14}$$

then (11) can be written as follows: Find $u \in H^1_\mu(K)$ such that

$$b(u,v) = \theta(v) \tag{15}$$

for any $v \in H^1_\mu(K)$. The existence and uniqueness of solutions to (15) can be proved using the classical Lax-Milgram Theorem.

We conclude this section by showing how our method in [11] may be used to the 1D eigenvalue-BVP in Eq. (11). Once again assuming that μ is non-atomic, we define the variable $t = F(x) = \mu((-\infty, x])$, where $F(x)$ denotes the cumulative distribution function associated with μ. Also let $x = F^{-1}(t)$. Using the change of variable presented in Proposition 2, we obtain

$$D^2_\mu u(x) + \lambda u(x) = \frac{d^2}{dt^2} u(F^{-1}(t))|_{t=F(x)} + \lambda u(x) = 0, \quad u(a) = 0,\ u(b) = 0\,,$$

which is equivalent to

$$\frac{d^2}{dt^2} u(F^{-1}(t))(t) + \lambda u(F^{-1}(t)) = 0\,.$$

By defining $g(t) = u(F^{-1})(t)$, this can be written as

$$\frac{d^2}{dt^2} g(t) + \lambda g(t) = 0, \quad g(0) = 0,\ g(1) = 0\,.$$

This, of course, is the classical "vibrating string" eigenvalue problem on $[0,1]$ with solutions $\lambda_n = (n\pi)^2$ and $g_n(t) = \sin(n\pi t)$, $n \geq 1$. From these, the solutions to (11) may expressed in terms of the variable x as simply $u_n(x) = \sin(n\pi F(x))$.

In each of Figures 1 - 3 are shown histogram approximations to the invariant measure μ and its CDF function F_μ along with the first three eigenfunctions $u_n(x)$. In Figure 1, the IFS is $w_1(x) = x/3$ and $w_2(x) = x/3 + 2/3$ with probabilities $p_1 = p_2 = 1/2$. This IFSP generates a "uniform" distribution on the classical middle-1/3 Cantor set. The same two IFS maps are employed in Figure 2, but with probabilities $p_1 = 2/5$ and $p_2 = 3/5$. The larger weight "towards the right" is evident in all portions of μ, F_μ (its CDF) and the eigenfunctions. In Figure 3, the two IFS maps are $w_1(x) = x/2$ and $w_2(x) = x/2 + 1/2$ with probabilities $p_1 = 2/5$ and $p_2 = 3/5$. Here, the attractor is $[0,1]$. Once again, the unequal weighting produces a (self-similar) "shift" of the measure to the right.

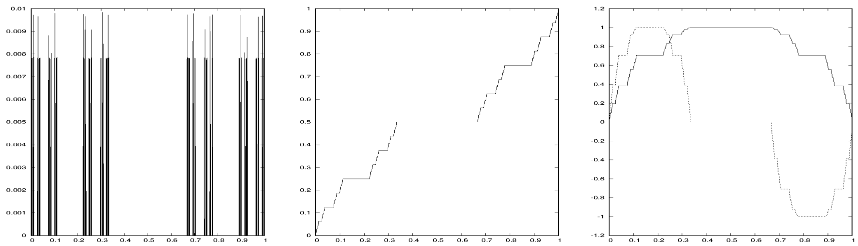

Fig. 1 "Uniform Cantor measure" μ, CDF F_μ, and first three eigenfunctions $u_n(x) = \sin(n\pi F_\mu(x))$.

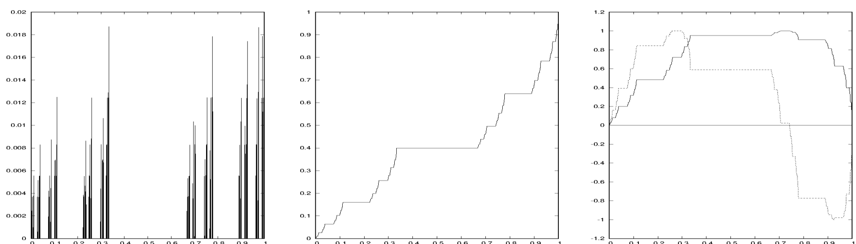

Fig. 2 "Non-uniform Cantor measure" μ, CDF F_μ, and first three eigenfunctions $\sin(n\pi F_\mu(x))$.

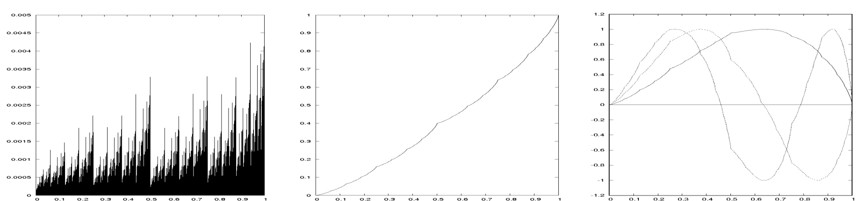

Fig. 3 Non-uniform fully supported measure μ, CDF F_μ, and first three eigenfunctions $\sin(n\pi F_\mu(x))$.

Note that in both Figure 1 and Figure 2 the eigenfunctions are illustrated by extending them to be constant over the "gaps" in the complement of the Cantor set. (These functions are supported only on the Cantor set itself.) This is done in order to make their graphs visible.

3 Some basics of Iterated Function Systems

In what follows, we let (X,d) denote a compact metric space. An N-map *Iterated Function System* (IFS) on X, $\mathbf{w} = \{w_1, \cdots, w_N\}$, is a set of N contraction mappings on X, i.e., $w_i : X \to X$, $i = 1, \cdots, N$, with contraction factors $c_i \in [0,1)$. (See [9, 1, 12].) Associated with an N-map IFS is the following set-valued mapping $\hat{\mathbf{w}}$ on the space $\mathscr{H}(X)$ of nonempty compact subsets of X,

$$\hat{\mathbf{w}}(S) := \bigcup_{i=1}^{N} w_i(S), \quad S \in \mathscr{H}(X). \tag{16}$$

Theorem 1. *[9] For $A, B \in \mathscr{H}(X)$,*

$$h(\hat{\mathbf{w}}(A), \hat{\mathbf{w}}(B)) \leq ch(A,B) \quad \textit{where} \;\; c = \max_{1 \leq i \leq N} c_i < 1 \tag{17}$$

and h denotes the Hausdorff metric on $\mathscr{H}(X)$.

Corollary 1. *[9] There exists a unique set $A \in \mathscr{H}(X)$, the* attractor *of the IFS* $\mathbf{w}$*, such that*

$$A = \hat{\mathbf{w}}(A) = \bigcup_{i=1}^{N} w_i(A). \tag{18}$$

Moreover, for any $B \in \mathscr{H}(X)$, $h(A, \hat{\mathbf{w}}^n B) \to 0$ as $n \to \infty$.

An N-map *Iterated Function System with Probabilities (IFSP)* $(\mathbf{w}, \mathbf{p})$ is an N-map IFS $\mathbf{w}$ with associated probabilities $\mathbf{p} = \{p_1, \cdots, p_N\}$, $\sum_{i=1}^{N} p_i = 1$. Let $\mathscr{M}(X)$ denote the set of probability measures on (Borel subsets of) X with Monge-Kantorovich distance d_{MK}: For $\mu, \nu \in \mathscr{M}(X)$,

$$d_{MK}(\mu, \nu) = \sup_{f \in Lip_1(X)} \left[\int f \, d\mu - \int f \, d\nu \right], \tag{19}$$

where $Lip_1(X) = \{ f : X \to \mathbb{R} \,|\, |f(x) - f(y)| \leq d(x,y) \}$. The metric space $(\mathscr{M}(X), d_{MK})$ is complete [9, 1].

Associated with an N-map IFSP is a mapping $M : \mathscr{M} \to \mathscr{M}$, often referred to as the *Markov operator*, defined as follows. Let $\nu = M\mu$ for any $\mu \in \mathscr{M}(X)$. Then for any measurable set $S \subset X$,

$$\nu(S) = (M\mu)(S) = \sum_{i=1}^{N} p_i \, \mu(w_i^{-1}(S)). \tag{20}$$

Theorem 2. *[9] For $\mu, \nu \in \mathscr{M}(X)$,*

$$d_{MK}(M\mu, M\nu) \leq c \, d_{MK}(\mu, \nu). \tag{21}$$

Corollary 2. *[9] There exists a unique measure $\bar{\nu} \in \mathscr{M}(X)$, the* invariant measure *of the IFSP $(\mathbf{w}, \mathbf{p})$, such that*

$$\bar{\mu}(S) = (M\bar{\mu})(S) = \sum_{i=1}^{N} p_i \bar{\mu}(w_i^{-1}(S)) \,. \tag{22}$$

Moreover, for any $\nu \in \mathscr{M}(X)$, $d_{MK}(\bar{\mu}, M^n \nu) \to 0$ as $n \to \infty$.

Theorem 3. *[9] The support of the invariant measure $\bar{\mu}$ of an N-map IFSP $(\mathbf{w}, \mathbf{p})$ is the attractor A of the IFS $\mathbf{w}$, i.e., supp $\bar{\mu} = A$.*

The next result is rather technical but is used in our convergence results in Section 4. The proof uses the fact that an IFSP on $\mathbb{R}$ induces a natural IFS-type operator on cumulative distribution functions which is contractive in the uniform norm.

Theorem 4. *Let $(\mathbf{w}, \mathbf{p})$ be an N-map IFSP with non-atomic invariant measure μ. Let $[a,b] = \mathrm{cov}(\mathrm{supp}(\mu))$ and suppose that $w_i([a,b]) \cap w_j([a,b])$ for $i \neq j$ either empty or consisting of one point.*

Let μ_0 be any initial Borel probability measure supported on $[a,b]$, $\mu_{n+1} = M\mu_n$, $F : [a,b] \to [0,1]$ be defined as $F(x) = \mu([a,x])$ and $F_n : [a,b] \to [0,1]$ be defined as $F_n(x) = \mu_n([a,x])$. Then $F_n \to F$ uniformly on $[a,b]$.

4 Convergence of Solutions

We now discuss a simple convergence result for the above eigenvalue problem. We restrict our presentation to the simplest case for clarity and brevity; more general results are certainly possible (including results on the variational problem (12)) but we leave them to our future paper.

Take an IFSP $(\mathbf{w}, \mathbf{p})$ and initial measure μ_0 so that they both satisfy the conditions of Theorem 4 and consider the sequence of eigenvalue problems: Find $u \in H^1_{\mu_n}([a,b])$ so that

$$\int_{[a,b]} D_{\mu_n} u D_{\mu_n} v d\mu_n + \lambda \int_{[a,b]} uv \, d\mu_n = 0 \,, \quad \text{for all } v \in H^1_{\mu_n}([a,b]) \,. \tag{23}$$

Proposition 4. *Given μ_n and u_n as above we have that $u_n \to u$ uniformly and u is solution to the problem:*

$$\int_{[a,b]} D_{\mu} u D_{\mu} v d\mu + \lambda \int_{[a,b]} uv d\mu = 0. \tag{24}$$

We end with a small taste of a more general problem. Start with μ_0 as the Lebesgue measure. Then the solutions u_n to the variational problems,

$$\int_{[a,b]} D_{\mu} u_n D_{\mu} v \, d\mu_n + \lambda \int_{[a,b]} u_n v \, d\mu_n = \int_{[a,b]} f v \, d\mu_n \,, \tag{25}$$

can be found by using more classical methods involving subproblems with weighted versions of the Lebesgue measure. Note that by using the definition of the Markov operator and a change of variable, the first term in (25) can be written as follows,

$$\int D_\mu u_n D_\mu v \, d\mu_n = \int D_\mu u_n D_\mu v \, dM^n \mu_0 =$$

$$\sum_{\sigma_1,\dots,\sigma_n=1}^{N} p_{\sigma_1} p_{\sigma_2} \cdots p_{\sigma_n} \int (D_\mu u_n D_\mu v) \circ w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n} \, d\mu_0.$$

Similarly,

$$\lambda \int uv \, d\mu_n = \sum_{\sigma_1,\dots,\sigma_n=1}^{N} p_{\sigma_1} p_{\sigma_2} \cdots p_{\sigma_n} \lambda \int (uv) \circ w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n} \, d\mu_0$$

and

$$\int fv \, d\mu_n = \sum_{\sigma_1,\dots,\sigma_n=1}^{N} p_{\sigma_1} p_{\sigma_2} \cdots p_{\sigma_n} \int (fv) \circ w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n} \, d\mu_0.$$

Thus the variational problem with respect to μ_n can be reformulated as follows: Find $u_n \in H^1_\mu([a,b]))$ such that

$$\sum_{\sigma_1,\dots,\sigma_n=1}^{N} p_{\sigma_1} p_{\sigma_2} \cdots p_{\sigma_n} \int_{K_s} (D_\mu u_n D_\mu v) \circ w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n} \, d\mu_0 +$$

$$\sum_{\sigma_1,\dots,\sigma_n=1}^{N} p_{\sigma_1} p_{\sigma_2} \cdots p_{\sigma_n} \lambda \int_{K_s} (uv) \circ w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n} \, d\mu_0 =$$

$$\sum_{\sigma_1,\dots,\sigma_n=1}^{N} p_{\sigma_1} p_{\sigma_2} \cdots p_{\sigma_n} \int (fv) \circ w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n} \, d\mu_0.$$

Notice that these integrals are all performed with respect to Lebesgue measure.

5 Conclusion

In [11] we introduced the notion of μ-derivative and we discussed how to formulate differential equations in which the derivative is replaced by a μ-derivative. In this paper, instead, we have extended this approach to the definition of weak derivative and to deal with boundary value problems. We have shown an application to a specific BVP, namely an eigenvalue problem, and presented a variational formulation of this problem in 1D. Future avenues include an extension of this approach to introduce weak partial derivatives to analyze variational problems on 2D fractals.

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