Chaos Games for Wavelet Analysis and Wavelet Synthesis

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Abstract

In this paper, we consider Chaos Games to generate wavelet decompositions and wavelet synthesis of functions. The primary technique involved is mixing the several Chaos Games for the individual wavelets in order to get the overall wavelet expansion. Generalizations to dilation factors other than two and to multidimensional wavelets are given.

Keywords: Wavelets, Iterated Function Systems, Chaos Game, Ergodic Theorem.

1 Introduction

There are many strong connections between the study of wavelets and fractals. For example, the scaling function satisfies a dilation equation (equation (8) which is a type of fractal self-similarity. Using this connection, Berger [2, 3] adapted the classical Chaos Game algorithm from IFS fractal theory to render the scaling function and mother wavelet. This allows one to compute function values for the mother wavelet ψ.

This paper starts with Berger's Chaos Game algorithm and modifies and extends it. The basic modification is to use the algorithm to compute integrals rather than function values. These integrals can then be used to compute the wavelet expansion coefficients for an arbitrary L^2 function f. This is the Wavelet Analysis.

On the other hand, by carefully "mixing" appropriate Chaos Game algorithms, we can start with the wavelet expansion coefficients and produce a rendering or approximation of the function f. This is the Wavelet Synthesis.

The organization of the paper is as follows. First, in sections 2-4 we give some necessary background on IFS fractals and wavelets. Then in section 5 we discuss the vectorized version of the dilation equation and the Chaos Game for generating the scaling function. In section 6 we modify this Chaos Game and obtain a Chaos Game for rendering the wavelets. Section 7 presents the parallel Chaos Games necessary to generate the coefficients in a wavelet expansion of a function while section 8 discusses "mixing" Chaos Games to generate a wavelet synthesis of a function. Finally, in section 9 we briefly discuss the extension of these ideas to more general dilation equations – to dilation equations with dilation factors other than 2 and to multidimensional dilation equations and wavelet expansions.

2 Iterated Function Systems

Here we briefly review the central concepts of IFS in order to provide a framework in which to the discuss the main issues of this article.

Let (X, d) be a compact metric space and $\{w_i\}_{i=1}^N$ be a finite collection of contractive maps with $w_i : X \to X$. The pair $(X, \{w_i\})$ is called an Iterated Function System (or **IFS**), though usually we refer to the maps alone as an IFS. Define the parallel action of the maps w_i by

$$\mathbf{w}(S) = \bigcup_{i=1}^{N} w_i(S). \tag{1}$$

Letting h denote the Hausdorff distance on the metric space X, we have ([1, 11])

Proposition 1 *There is a unique compact set $A \subset X$, the* attractor of the IFS, *such that*

$$A = \bigcup_{i=1}^{N} w_i(A) \tag{2}$$

and $h(\mathbf{w}^n(S), A) \to 0$ as $n \to \infty$, for every non-empty compact set $S \subset X$.

If we associate with each w_i a non-zero probability, p_i, then the triad $\{X, \{w_i\}_{i=1}^N, \{p_i\}_{i=1}^N\}$ is known as an Iterated Function System with Probabilities, or IFSP.

Let $\mathcal{B}$ be the $\sigma - algebra$ of Borel subsets of X and let $\mathcal{P}$ be the set of all probability measure on X. The Monge-Kantorovich metric (often called the Hutchinson metric in the IFS literature) on $\mathcal{P}$ is defined as

$$d(\mu, \nu) = \sup_{f \in Lip_1} \{\int_X f d\mu - \int_X f d\nu\} \quad \text{for all } \mu, \nu \in \mathcal{P} \tag{3}$$

where $Lip_1 = \{f : X \to I\!R \ : \ |f(x) - f(y)| \leq d(x, y)\}$ is the set of Lipschitz functions with Lipschitz factor 1.

Now we define the following Markov operator on the space $\mathcal{P}$,

$$M(\nu) = \sum_{i=1}^{N} p_i \nu \circ w_i^{-1}. \tag{4}$$

For this operator we have the following proposition (see [1, 11]).

Proposition 2 *There is a unique measure $\mu \in \mathcal{P}$, called the* invariant measure, *such that*

$$M(\mu) = \mu \tag{5}$$

Further, supp$(\mu) = A$

3 Rendering and approximating attractors of IFS

There are two basic algorithms for generating the attractor of a given IFS. One is a deterministic algorithm, the other is a stochastic algorithm, known as the Chaos Game (see [1]).

Given an IFS with N contractive maps $\{w_i\}_{i=1}^N$, we may approximate the attractor deterministically as follows:

1. Pick $x_0 \in X$ and form the set $S_0 = \{x_0\}$ and fix an $\epsilon > 0$
2. Form the set $S_n = \mathbf{w}(S_{n-1})$,
3. If $h(S_n, S_{n-1}) \geq \epsilon$, goto step 2.
4. Plot S_n.

This algorithm is essentially computing the fixed the point of the set-valued function $\mathbf{w}$ through a direct iteration process. At the n^{th} iteration there are N^n points. When rendering attractors on a computer screen the algorithm could terminate once resolution of the screen prevents any further detail from being revealed; it is this consideration which typically sets ϵ.

Given an IFSP with N contractive maps $\{w_i\}_{i=1}^N$, with associated probabilities $\{p_i\}_{i=1}^N$ we may approximate the attractor stochastically as follows:

1. Pick $x_0 \in X$ to be the fixed point of w_1 (this guarantees that all points determined henceforth in this algorithm will lie on the attractor).
2. Pick $\sigma_n \in \{1, \ldots, N\}$ at random according to the probabilities $\{p_i\}_{i=1}^N$ and set $x_n = w_{\sigma_n}(x_{n-1})$
3. Plot x_n
4. If n is less than the maximum number of iterations, go to step 2.

Elton [9] proved the following theorem, commonly known as Elton's Ergodic Theorem, related to the Chaos Game.

Proposition 3 *Let $\{X, \{w_i\}_{i=1}^N, \{p_i\}_{i=1}^N\}$ be an IFSP with invariant measure μ. Then for every starting point $x_0 \in X$ and almost every sequence $\{\sigma_n\}$ generated by the Chaos Game algorithm, we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(x_k) = \int_X f(x)\ d\mu(x) \tag{6}$$

for all continuous and simple functions $f : X \to \mathbb{R}$.

Suppose that we set $f = \chi_B$, the characteristic function of $B \in \mathcal{B}$. Then from Elton's theorem,

$$\lim_{n\to\infty} \frac{\#\{x_n \in B\}}{n} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_B(x_k) = \mu(B). \tag{7}$$

This explains why the Chaos Game generates an approximation to the attractor. For each set $B \subset X$ so that $\mu(B) > 0$, we know that eventually $x_n \in B$ so we will eventually plot a point in B. Thus, the Chaos Game will eventually plot each "pixel" in A.

4 Necessary wavelets background

In this section we discuss those parts of wavelet theory that will be needed for the purposes of this paper. For a more complete introduction, see the very nice book [7].

Let ϕ be a scaling function for a MultiResolution Analysis. Then $\{\phi(\cdot - i) | i \in \mathbf{Z}\}$ is an orthonormal set and ϕ satisfies a dilation equation of the form

$$\phi(x) = \sum_{n\in\mathbf{Z}} h_n \phi(2x - n). \tag{8}$$

We will assume that the only non-zero coefficients h_n are $h_0, h_1, \ldots, h_N$. In this case, it can be shown that ϕ is supported on the interval $[0, N]$.

Associated with the scaling function ϕ there is a function ψ, called the *mother wavelet*, which is defined by

$$\psi(x) = \sum_k (-1)^k h_{N-k} \phi(2x - k). \tag{9}$$

One can show that ψ is also supported on $[0, N]$ with this definition. Define $\psi_{i,j}(x) = 2^{i/2}\psi(2^i x - j)$. Then it turns out that the set $\{\psi_{i,j} \mid i, j \in \mathbf{Z}\}$ forms an orthonormal basis for $L^2(\mathbb{R})$. The important feature for us is that this basis is generated by the dilations and translations of a single function – the mother wavelet ψ.

5 Rendering a Compactly Supported Scaling Function

In general, a compactly supported scaling function as defined above does not have a closed analytic form. In the following we will discuss an IFS type algorithm to approximate the graph of ϕ to arbitrary accuracy.

Let us suppose that we have chosen a sequence of h_n's, subject to the our assumption that $h_n = 0$ for $n < 0$ and $n > N$, in such a way that ϕ induces a multiresolution analysis on $L^2(\mathbb{R})$. Daubechies and Lagarias [8] and Micchelli and Prautzsch [13] independently noticed that one could vectorize the dilation equation. Define $V_\phi : [0,1] \to \mathbb{R}^N$ as

$$V_\phi(x) = \begin{pmatrix} \phi(x) \\ \phi(x+1) \\ \vdots \\ \phi(x+N-1) \end{pmatrix} \tag{10}$$

Define the $N \times N$ matrices T_0 and T_1 by $(T_0)_{i,j} = h_{2i-j-1}$ and $(T_1)_{i,j} = h_{2i-j}$. That is,

$$T_0 = \begin{pmatrix} h_0 & 0 & 0 & \cdots & 0 & 0 \\ h_2 & h_1 & h_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & h_N & h_{N-1} \end{pmatrix}$$

and

$$T_1 = \begin{pmatrix} h_1 & h_0 & 0 & \cdots & 0 & 0 \\ h_3 & h_2 & h_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & h_N \end{pmatrix}.$$

Let $\tau : [0,1] \to [0,1]$ be defined as $\tau(x) = 2x \bmod 1$. Then the dilation equation becomes (assuming that ϕ is continuous)

$$V_\phi(x) = T_\omega V_\phi(\tau x) \tag{11}$$

where ω is the first digit in the binary expansion of x.

As an example, we compute the transformation T_0 corresponding to the Daubechies-4 wavelet. In this case only h_0, h_1, h_2, and h_3 are non-zero, so ϕ is supported on the interval $[0,3]$. Thus

$$V_\phi(x) = \begin{pmatrix} \phi(x) \\ \phi(x+1) \\ \phi(x+2) \end{pmatrix} \tag{12}$$

From the dilation equation, we have the equations

$$\begin{array}{lcllllllll}
\phi(x) & = & h_0\phi(2x) & + & h_1\phi(2x-1) & + & h_2\phi(2x-2) & + & h_3\phi(2x-3) \\
\phi(x+1) & = & h_0\phi(2x+2) & + & h_1\phi(2x+1) & + & h_2\phi(2x) & + & h_3\phi(2x-1) \\
\phi(x+2) & = & h_0\phi(2x+4) & + & h_1\phi(2x+3) & + & h_2\phi(2x+2) & + & h_3\phi(2x+1).
\end{array}$$

After rejecting the contributions from outside the support of ϕ it follows that (for $0 < x < 1/2$)

$$\begin{pmatrix} \phi(x) \\ \phi(x+1) \\ \phi(x+2) \end{pmatrix} = \begin{pmatrix} h_0 & 0 & 0 \\ h_2 & h_1 & h_0 \\ 0 & h_3 & h_2 \end{pmatrix} \begin{pmatrix} \phi(2x) \\ \phi(2x+1) \\ \phi(2x+2) \end{pmatrix}.$$

Clearly the matrix is T_0. The computations for $x > 1/2$ are similar.

In the papers [2, 3], Berger first noticed that this IFS approach to the dilation equation naturally leads to a Chaos Game. We now set up a vectorized IFSP. The graph of the scaling function ϕ can be approximated through the following modified version of the Chaos Game.

1. Initialize $x = 0$, and the vector $V_\phi(x)$ to be the fixed point of T_0
2. Pick $\omega \in \{0, 1\}$ with equal probability.
3. $x \mapsto x/2 + \omega/2$
4. $V_\phi(x) \mapsto T_\omega V_\phi(x)$
5. Plot the points $(x, \phi(x)), (x+1, \phi(x+1)), \ldots, (x+N-1, \phi(x+N-1))$
6. If number of iterations is less than maximum, go to step 2.

The basic reason that this simple algorithm works is that the vectorized IFS is non-overlapping, that is that $m(w_0([0,1]) \bigcap w_1([0,1])) = 0$. See [10] for more discussion of Chaos Games for IFS on functions.

6 Modified Chaos Game Algorithm for Wavelet Generation

In this section we modify the Chaos Game which generates the scaling function, ϕ, in order to generate the wavelet ψ. This algorithm is a slight variation of Berger's algorithm given in [2, 3] in that we generate a piecewise constant approximation to ψ by calculating the average value of ψ on some partition.

The algorithm is as follows. Let B_i be a partition of $[0, N]$ into Borel sets and for each B_i we have S_i, an accumulation variable. Let $x = 0$ and $y \in \mathbb{R}^N$ be the (normalized) fixed point of T_0. Initialize $S_j = 0$.

1. Choose $\omega \in \{0, 1\}$ with equal probability.

2. $x \mapsto x/2 + \omega/2$

3. $y \mapsto T_\omega(y)$

4. for each $k = 0, 1, \ldots, 2N-1$,

   - if $x/2 + k/2 \in B_m$, then update

$$S_m + = \sum_{i=max(0,k-N+1)}^{min(N,k)} (-1)^i h_{N-i} y_{k-i}$$

5. If number of iterations is less than maximum, go to step 1.

The approximation to ψ on B_m is

$$\frac{S_m}{2 \times m(B_m) \times numits} \tag{13}$$

Basically what this algorithm does is generate the scaling function via the previous Chaos Game and take the appropriate dilation and linear combinations (from equation (9)) in order to form ψ.

The extra factor of 2 in the denominator is because ψ is defined in terms of $\phi(2x - i)$ rather than $\phi(x)$.

Another way to write step 4 of the algorithm is the following. We have the values $y_j = \phi(x + j)$ for $j = 0, 1, \ldots, N-1$ and we have the coefficients h_i for $i = 0, 1, \ldots, N$. For $z = (x + i + j)/2$, we know that $\psi(z)$ depends on the term $(-1)^i h_{N-i} y_j$. So, for each i, j so that $(x + i + j)/2 \in B_m$, we update

$$S_m + = (-1)^i h_{N-i} y_j.$$

In our algorithm in step 4, we simply group all those i, j so that $i + j = k$ and loop through k.

Proposition 4 *For each m*

$$\frac{S_m}{2 \times numits \times m(B_m)} \to \frac{1}{m(B_m)} \int_{B_m} \psi(x)\ dx \qquad as \qquad numits \to \infty.$$

Proof: The process is a non-overlapping vector valued IFSM on $[0,1]$ (see [10]). We map the vector valued function $g : [0,1] \to I\!R^N$ to the function $\hat{g} : [0, N] \to I\!R$ by

$$\hat{g}(x) = \sum_{i=1}^{N} \chi_{[i-1,i]} g(x)_i.$$

Through this mapping, the process is equivalent to a non-overlapping IFSM on $[0, N]$. Now at each stage y represents

$$(\phi(x), \phi(x+1), \ldots, \phi(x+N-1)).$$

To make the notation simpler, we just set $\phi(x+i)=0$ if $i<0$ or $i>N-1$ in the formulas below. So letting x_n be the orbit generated by the Chaos Game, we have

$$
\begin{aligned}
\frac{S_m}{numits \times 2} &= \frac{1}{2 \times numits} \sum_n \sum_{k=0}^{2N-1} \sum_{i=0}^{N} (-1)^i h_{N-i} \phi(x_n + k - i) \chi_{B_m}\left((x_n+k)/2\right) \\
&\longrightarrow 1/2 \int_0^1 \sum_{k=0}^{2N-1} \sum_{i=0}^{N} (-1)^i h_{N-i} \phi(x+k-i) \chi_{B_m}\left((x+k)/2\right)\ dx \\
&= \sum_{k=0}^{2N-1} \int_{k/2}^{(k+1)/2} \sum_{i=0}^{N} h_{N-i} \phi(2y-i) \chi_{B_m}(y)\ dy \\
&= \int_0^N \sum_i h_{N-i} \phi(2y-i) \chi_{B_m}(y)\ dy \\
&= \int_{B_m} \psi(y)\ dy.
\end{aligned}
$$

Thus, we have the desired result. ∎

Notice that $1/m(B_m)\int_{B_m} \psi(x)\ dx$ is the average value of ψ over B_m. Define

$$\bar{\psi}(x) = \sum_m 1/m(B_m) \times \chi_{B_m}(x) \times \int_{B_m} \psi\ dx. \tag{14}$$

If we refine the partition B_j, we will get a refined estimate of ψ. The next proposition shows that in the limit, we recover ψ exactly.

Proposition 5 *Let $\mathcal{P}_n$ be a nested sequence of Borel partitions of $[0,N]$ whose "sizes" go to zero as $n \to \infty$. Let $\bar{\psi}_n$ be the average value function of ψ associated with $\mathcal{P}_n$. Then $\bar{\psi}_n$ converges to ψ pointwise almost everywhere.*

Proof: Just notice that $\bar{\psi}_n$ is the conditional expectation of ψ given $\mathcal{P}_n$. Since $\psi \in L^1$ (ψ is a continuous compactly supported function), we have the result by the Martingale Convergence Theorem ([14]). ∎

Suppose that we want to generate the wavelet $\psi_{i,j}$ instead of the mother wavelet ψ. The only change to the above algorithm necessary is to replace Step 4 with

- if $w_{i,j}(x/2+k/2) \in B_m$, then update

$$S_m+ = 2^{i/2} \sum_{l=max(0,k-N+1)}^{min(N,k)} (-1)^l h_{N-l} y_{k-l}$$

and then the approximation to $\psi_{i,j}$ on B_m is

$$\frac{S_m}{2 \times numits \times m(w_{i,j}^{-1}(B_m))} = \frac{S_m}{numits \times m(B_m) \times 2^{i+1}}$$

The scaling factor $2^{i/2}$ is necessary since $\psi_{i,j}(x) = 2^{i/2}\psi(2^i x - j)$.

With the modified Step 4, we have

$$\frac{S_m}{2 \times numits \times m(w_{i,j}^{-1}(B_m))} \to 1/m(B_m) \int_{B_m} \psi_{i,j}(x)\ dx \tag{15}$$

since

$$1/m(B_m) \int_{B_m} \psi_{i,j}(x)\ dx = \frac{1}{2^i m(B_m)} \int_{w_{i,j}^{-1}(B_m)} 2^{i/2}\psi(x)\ dx.$$

7 Chaos Game for Wavelet Analysis

Given an f in L^2, we can represent

$$f = \sum c_{i,j}\psi_{i,j}$$

since the wavelets $\psi_{i,j}$ form an orthonormal basis. This is the wavelet analysis of f.

Suppose we modify the algorithm of the previous section by replacing step 4 with the following:

$$S_m+ = f(x/2+k/2) \times \sum_{i=max(0,k-N+1)}^{min(N,k)} (-1)^i h_{N-i} y_{k-i} \tag{16}$$

Then, by similar arguments to those in the proof of Proposition 4, we would have that

$$\frac{S_m}{2 \times numits \times m(B_j)} \to \frac{1}{m(B_j)} \int_{B_j} f(x)\psi(x)\ dx \quad \text{as} \quad numits \to \infty. \tag{17}$$

Now, in wavelet analysis, we wish to compute integrals of the form

$$c_{i,j} = \int f(x)\psi_{i,j}(x)\ dx.$$

So, we now discuss how to modify the algorithm of the previous section in such a way as to do this in parallel for many different pairs (i, j). We use as motivation equation (17).

Suppose that we want to calculate $c_{i,j}$ for $(i, j) \in \Lambda$, some collection of coefficients. For each $\sigma \in \Lambda$, let S_σ be an accumulation variable, initialized to 0. Let

$$w_\sigma(x) = w_{i,j}(x) = x/2^i + j/2^i,$$

the map that maps the mother wavelet ψ to the wavelet $\psi_{i,j}$ (in that $\psi_{i,j} = 2^{i/2}\psi \circ w_{i,j}^{-1}$). For $\sigma = (i, j) \in \Lambda$, define $scale(\sigma) = i$.

Our modified algorithm is as follows:

1. Initialize $x = 0$ and $y \in I\!R^N$ to be the fixed point of T_0.
2. Choose $\omega \in \{0, 1\}$ with equal probability.
3. $x \mapsto x/2 + \omega/2$
4. $y \mapsto T_\omega(y)$
5. For each $\sigma \in \Lambda$, we do
   - for each $k = 0, 1, \ldots 2N - 1$
     - Update

$$S_\sigma += f(w_\sigma(x/2 + k/2)) \times \sum_{i=max(0,k-N+1)}^{min(N,k)} (-1)^i h_{N-i} y_{k-i}$$

6. If number of iterations is less than maximum, go to step 2.

The approximation to c_σ is

$$\frac{S_\sigma}{2 \times 2^{scale(\sigma)} \times numits}. \tag{18}$$

Proposition 6 *For each $\sigma \in \Lambda$*

$$\frac{S_\sigma}{2 \times 2^{scale(\sigma)} \times numits} \to \int f(x)\psi_\sigma(x)\ dx \quad \text{as} \quad numits \to \infty.$$

Proof: The proof is similar to the proof of Proposition 4. In this case, we are evaluating the function $g(x) = \psi_\sigma(x) f(x)$ along the trajectory. Thus,

$$\frac{S_\sigma}{2 \times 2^{scale(\sigma)} \times numits} \to \int \psi_\sigma(x) f(x)\ dx$$

which is what we wished to show. ∎

What if we do not have the true function f but only have some approximation to f? The typical case would be where we have a piece-wise constant approximation to f (as in the case of a function that we only sample at certain values or a function which is the result of a Chaos Game approximation). Call this approximation $\hat{f}$. Then what we can compute is

$$\hat{c}_{i,j} = \int \hat{f}(x)\psi_{i,j}(x)\ dx.$$

As long as we know that $\hat{f}$ is close to f, then we know that $\hat{c}_{i,j}$ is close to $c_{i,j}$.

A special case of potential interest is when the function f is also the attractor for a related IFS. Then we can run two Chaos Games simultaneously to obtain a wavelet decomposition of f.

8 Chaos Game for Wavelet Synthesis

Let

$$f = \sum_{i,j} c_{i,j}\psi_{i,j}$$

be the decomposition of a function in a wavelet basis. We wish to recover the function f from the coefficients $c_{i,j}$. For the moment, we assume that there are only finitely many non-zero coefficients. Let $\Lambda = \{(i,j)|c_{i,j} \neq 0\}$.

Since Λ is finite, the support of f is compact, say the interval $[A, B]$.

Let $E = \sum_{i,j} |c_{i,j}|$ and $p_{i,j} = |c_{i,j}|/E$. Using the notation from previous sections, we have the following algorithm.

Initialize $x = 0$ and $y \in \mathbb{R}^N$ to be the fixed point of T_0. Let B_j be a partition of $[A, B]$ into Borel sets with associated accumulation variables S_j initialized to be 0.

Let $w_{i,j}(x) = x/2^i + j/2^i$, the map that maps the mother wavelet ψ to the wavelet $\psi_{i,j}$.

1. Choose $\omega \in \{0, 1\}$ with equal probability.
2. $x \mapsto x/2 + \omega/2$
3. $y \mapsto T_\omega(y)$
4. Choose $(i,j) \in \Lambda$ with probability $p_{i,j}$.
5. For $k = 0, 1, \ldots, 2N - 1$
   - if $w_{i,j}(x/2 + k/2) \in B_m$, then update

$$S_m += SGN(c_{i,j})\frac{2^{i/2}}{2^i}\left(\sum_{l=max(0,k-N+1)}^{min(N,k)} (-1)^l h_{N-l}y_{k-l}\right)$$

6. If number of iterations is less than maximum, go to step 1.

The approximation to f on B_m is

$$\frac{E \times S_m}{2 \times m(B_m) \times numits} \tag{19}$$

Proposition 7 *For each* m

$$\frac{S_m}{2 \times numits \times m(B_m)} \to \frac{1}{E \times m(B_m)} \int_{B_m} f(x)\ dx \quad \text{as} \quad numits \to \infty.$$

Proof: For each $\sigma \in \Lambda$, let S_m^σ be an accumulation variable corresponding to B_m. Suppose we modify the above algorithm so that when we are in state σ, we modify only S_m^σ. Then, by Proposition 4, we know that

$$\frac{S_m^\sigma}{2 \times numits \times m(B_m)} \to SGN(c_{i,j})p_\sigma/m(B_m) \int_{B_m} \psi_\sigma(x)\ dx.$$

Now,

$$S_m = \sum_{\sigma \in \Lambda} S_m^\sigma$$

so

$$\frac{S_m}{2 \times numits \times m(B_m)} = \sum_{\sigma \in \Lambda} \left(\frac{S_m^\sigma}{2 \times numits \times m(B_m)} \right) \to \sum_{\sigma \in \Lambda} SGN(c_{i,j})p_\sigma/m(B_m) \int_{B_m} \psi_\sigma(x)\, dx$$

and

$$\sum_{\sigma \in \Lambda} SGN(c_{i,j})p_\sigma/m(B_m) \int_{B_m} \psi_\sigma(x)\ dx = 1/m(B_m)/E \int_{B_m} f(x)\ dx.$$

Thus, we have proved the proposition. ▮

As in the case of the Chaos Game for ψ, we have that if we refine the partition B_m, we will get a refined estimate of ψ. For completeness, we record the next proposition. The proof is just like the proof of Proposition 5.

Proposition 8 *Let $\mathcal{P}_n$ be a nested sequence of Borel partitions of $[A, B]$ whose "sizes" go to zero as $n \to \infty$. Let $\bar{f}_n$ be the average value function of f associated with $\mathcal{P}_n$. Then $\bar{f}_n$ converges to f pointwise almost everywhere.*

9 Extensions

9.1 Arbitrary function in $L^2(\mathbb{R})$

Let us take $f \in L^2(\mathbb{R})$, then in the wavelet basis

$$f = \sum_{ij} c_{ij} \psi_{ij}.$$

If we fix an $\epsilon > 0$, then there is a finite subset $\mathcal{F}$ of $\mathbf{Z}$ so that if we define the "truncation" of f, f_{tr}, as

$$f_{tr} = \sum_{(i,j) \in \mathcal{F}} c_{ij} \psi_{ij}$$

it follows that

$$\|f - f_{tr}\|_{L^2} < \frac{\epsilon}{3}.$$

Notice that f_{tr} is a finite linear combination of compactly supported functions, and consequently must be compactly supported itself. Let B_k be a Borel partition of $supp(f_{tr})$, and let $\bar{f}$ be the average value function associated with this partition (see equation 14). In addition let us suppose that that partition B_k is sufficiently refined that

$$\|f_{tr} - \bar{f}\|_{L^2} < \frac{\epsilon}{3}$$

Suppose we generate a Chaos Game approximation to the function f_{tr}, call it f_{cg}. According to Proposition 7 by choosing a maximal number of Chaos Game iterations sufficiently large then

$$\|\bar{f} - f_{cg}\|_{L^2} < \frac{\epsilon}{3}$$

almost certainly.

This in turn implies that

$$\|f - f_{cg}\|_{L^2} \quad \leq \quad \epsilon$$

which proves the following proposition.

Proposition 9 *Let $f \in L^2(\mathbb{R})$, and $\epsilon > 0$, then there exists a Chaos Game defined by the algorithm in section 8 which produces a function f_{cg} which almost surely satisfies*

$$\|f - f_{cg}\|_{L^2} < \epsilon$$

9.2 General dilations and higher dimensions

The results in this paper can be extended to more general dilation equations and more general wavelets.

For dilation factor N (instead of 2), we simply get maps $T_0, T_1, \ldots, T_{N-1}$ corresponding to the N map vector IFSM

$$\mathcal{T}(f) = \sum_{i=0}^{N-1} T_i f(Nx - i)$$

with f being the vectorized version of the scaling function as usual. Then the Chaos Game for the scaling function is a simple modification of our Chaos Game from section 5. Once we have a Chaos Game for the scaling function, the Chaos Game for the wavelet and for wavelet analysis and wavelet synthesis follow.

In order to extend our results to multidimensional scaling functions (that is, $\phi : \mathbb{R}^n \to \mathbb{R}$), wavelets and wavelet expansions, we need a formulation of the vector IFS in this, more general, context.

We briefly indicate the theory here. For a more complete discussion see the references [4, 5, 6, 12, 15]

Given a *dilation matrix* $A \in M_n(\mathbf{Z})$ and a set of coefficients c_n, we have the dilation equation

$$\phi(x) = \sum_{n \in X} c_n \phi(Ax - n) \tag{20}$$

where $X \subset \mathbf{Z}^n$ is a finite set. The scaling function will be the solution to this equation.

For $\Omega \subset \mathbf{Z}^n$, we let K_Ω be the attractor of the IFS $W_\Omega = \{A^{-1}x + A^{-1}d \, : \, d \in \Omega\}$. Then if ϕ is a compactly supported solution to equation (20) $supp(\phi) \subset K_X$.

In one dimension, the vectorized function V_ϕ is based covering the support of the scaling function with copies of $[0, 1]$. In $\mathbb{R}^n$, we need to base the vectorized function on a self-affine tiling of $\mathbb{R}^n$. Under certain conditions (see [15]), there is a *digit set* $\mathcal{D} \subset \mathbf{Z}^n$ (necessarily a set of complete representatives of $\mathbf{Z}^n / AZ^n$) so that $K_\mathcal{D}$ is a tile for $\mathbb{R}^n$ (i.e. that $Z^n + K_\mathcal{D}$ tiles $\mathbb{R}^n$).

So, we convert the dilation equation (20) into the vectorized equation

$$V_\phi(x) = T_\omega V_\phi(\tau x). \tag{21}$$

Here $V_\phi : K_\mathcal{D} \to \mathbb{R}^S$, where $S \subset Z^n$ is a "covering set" (see [12], pg 85), and for each $\omega \in \mathcal{D}$ we have T_ω is an $|S| \times |S|$ matrix defined by $(T_\omega)_{m,n} = c_{\omega + Am - n}$ for $m, n \in S$ and $\tau : K_\mathcal{D} \to K_\mathcal{D}$ is the shift map (see [12], pg 83).

This gives a nonoverlapping vector IFS, so we can use the Chaos Game to generate the attractor. The following algorithm will accomplish this (assuming that the dilation equation has a solution).

Given $c_n \, : \, n \in X \subset \mathbf{Z}^n$, the dilation matrix A, $\mathcal{D} \subset \mathbf{Z}^n$ a digit set for a tiling based on A, $S \subset \mathbf{Z}^n$ a covering set. Let B_n be a partition of compact sets so that $K_X \subset \bigcup B_i$. For each B_i, let S_i be an accumulation variable initialized to 0.

1. Choose $d \in \mathcal{D}$ and initialize x as the fixed point of the map $x \to A^{-1}x + A^{-1}d$. Initialize $y \in \mathbb{R}^S$ as the fixed point of T_d
2. Choose $\omega \in \mathcal{D}$ randomly (with equal probability).
3. Update $y = T_\omega y$ and $x = A^{-1}x + A^{-1}\omega$.
4. For each $m \in S$, if $x + m \in B_i$ update
$$S_i \mathrel{+}= \ y_m.$$
5. If the number of iterations is less than the maximum, go to step 2.

The approximation to ϕ on B_i is

$$\frac{S_i}{m(B_i) \times numits} \tag{22}$$

and the proof of the convergence to the average value of ϕ on B_i is the same as in the one dimensional case.

It is relatively straightforward to modify this basic algorithm to obtain a Chaos Game algorithm for generating the mother wavelet, for doing wavelet analysis or for doing wavelet synthesis.

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