

Set-Valued Φ -Coherent Risk Measures for Sustainable Decision-Making*

Davide La Torre^{1*}, Franklin Mendivil² and Matteo Rocca³

^{1*}SKEMA Business School, Université Côte d’Azur, Sophia Antipolis, France.

²Department of Mathematics and Statistics, Acadia University, Wolfville, Nova Scotia, Canada.

³Department of Economics, Università degli Studi dell’Insubria, Varese, Italy.

*Corresponding author(s). E-mail(s): davide.latorre@skema.edu;
Contributing authors: franklin.mendivil@acadiau.ca;
matteo.rocca@uninsubria.it;

Abstract

Coherent risk measures, with their rigorous mathematical foundation, are critical for assessing and managing risks in fields such as sustainability, where uncertainty, incomplete data, and multidimensional risks pose significant challenges. This paper extends the concept of coherent risk measures by incorporating set-valued probabilities, which represent uncertainty as a family of probability measures, enabling robust risk evaluation under incomplete information. The proposed framework also accommodates multidimensional risks, making it particularly suited for addressing complex sustainability challenges.

Keywords: Set-Valued Risk Measure, Set-Valued Probability Measure, Risk Minimization, Sustainable Decision Making

1 Introduction

Risk assessment is a critical aspect of decision-making in a wide range of fields, including finance, engineering, and environmental sustainability. The concept of a *coherent*

* All authors declare they have no conflict of interest.

risk measure has emerged as a rigorous and mathematically sound framework for quantifying risk. Coherent risk measures, first introduced in the context of finance, provide four essential properties—*monotonicity*, *sub-additivity*, *positive homogeneity*, and *translation invariance*—that ensure consistency and robustness in risk evaluations. These properties make coherent risk measures particularly attractive for applications where risk must be assessed, minimized, and managed systematically.

While coherent risk measures were initially developed to address financial risks, their applicability extends far beyond this domain. In the context of sustainability, risk assessment often involves evaluating environmental, social, and operational risks that may threaten the resilience and long-term viability of systems. For example, sustainability risks include exceeding greenhouse gas (GHG) emissions targets, depleting natural resources, or crossing planetary boundaries related to biodiversity or water use. These risks are inherently uncertain and often involve extreme events, making the need for robust and reliable risk measures paramount. Coherent risk measures, with their emphasis on worst-case scenarios and their ability to account for diversification, are well-suited to address the complexities of sustainability risk assessment.

In sustainability, the stakes are high, and the consequences of poor risk management can be severe. Therefore, it is essential to use definitions and methods that are robust and reliable. Traditional risk measures often rely on precise probability distributions, which may not be available or reliable in real-world sustainability applications. For example when dealing with environmental applications, such as GHG emissions or biodiversity loss, data are often incomplete or uncertain, making it difficult to assign precise probabilities to outcomes. Furthermore, sustainability challenges involve multiple, interrelated risks (e.g., climate change, resource depletion, and social inequality), which require a framework capable of handling multidimensional and uncertain data. To address these challenges, it is necessary to extend the classical notion of coherent risk measures to frameworks that are more robust and flexible. One promising approach is to use *set-valued probabilities*, which allow for uncertainty in the underlying probability distributions. Set-valued probabilities represent uncertainty as a family of probability measures, rather than a single precise measure, enabling the model to capture a wider range of possible scenarios. This approach enhances robustness and ensures that the resulting risk measure is more reliable, even under uncertain or incomplete information.

The use of set-valued probabilities provides a natural extension of coherent risk measures. In this framework, the risk measure is defined not with respect to a single probability measure but with respect to a set-valued probability. This framework allows to describe a family of probability measures with a single object.

It is well known that a coherent risk measure $\rho(X)$ for a random variable X can be expressed in its dual form as:

$$\rho(X) = \sup_{Q \in \mathcal{Q}} \mathbb{E}_Q[-X],$$

where $\mathcal{Q}$ represents the set of plausible probability measures under consideration. The notion of coherent risk measure with respect to a set-valued probability is essentially

based on the dual form of a canonical risk measure in which the family $\mathcal{Q}$ is replaced by a set-valued probability.

Set-valued probabilities also enable the extension of coherent risk measures to multidimensional settings, where risks involve multiple criteria, such as economic, environmental, and social dimensions. In such cases, the risk measure $\rho(X)$ may map to a set of outcomes, reflecting the trade-offs and uncertainties inherent in sustainability decision-making.

The paper is organized as follows. Section 2 recalls the basic definitions of set-valued functions. Section 3 presents the notion of set-valued probability and its main properties. Section 4 introduces the notion of coherent risk measure. Section 5 presents the notion of set-valued Φ -coherent risk measure. Section 6 presents the risk minimization problem with respect to a Φ -coherent risk measure as well as optimality conditions and stability results. Section 7 is dedicated to an illustrative example in the domain of sustainable decision making and, finally, Section 8 concludes.

2 Set-Valued Functions

We now provide a very brief introduction to some elements of the theory of convex sets and set-valued functions. We invite the reader to consult with [2, 4, 13] for more details. Throughout, $\mathcal{K}$ denotes the collection of all nonempty, compact, and convex subsets of $\mathbb{R}^d$. Elements of $\mathcal{K}$ can be added, or we can perform scalar multiplication on them, by using

$$A + B := \{a + b : a \in A, b \in B\} \text{ and } \lambda A = \{\lambda a : a \in A\}, \lambda \in \mathbb{R}.$$

It is important to note that $(\lambda_1 + \lambda_2)A = \lambda_1 A + \lambda_2 A$ is true as long as $\lambda_1 \lambda_2 \geq 0$ but not if $\lambda_1 \lambda_2 < 0$. This points out the important limitation that subtraction does not work as we might like.

An element $A \in \mathcal{K}$ is *nonnegative* ($A \geq 0$) if $0 \in A$. The *support function* $spt(\cdot, A) : \mathbb{R}^d \rightarrow \mathbb{R}$ for $A \in \mathcal{K}$ is given by

$$spt(p, A) = \sup\{p \cdot a : a \in A\},$$

which completely defines A since

$$A = \bigcap_{\|p\|=1} \{x : x \cdot p \leq spt(p, A)\}. \quad (1)$$

It is important to note that $A \geq 0$ iff $spt(p, A) \geq 0$ and, more generally, $A \subseteq B$ if and only if $spt(p, A) \leq spt(p, B)$ for all $\|p\| = 1$. In addition, for any $\lambda \geq 0$ and $A, B \in \mathcal{K}$, we have that

$$spt(p, \lambda A + B) = \lambda spt(p, A) + spt(p, B), \quad spt(p, -B) = spt(-p, B).$$

Normally, however, $spt(p, -A) \neq -spt(p, A)$, and so the support functions is almost but not quite linear.

For any $A \in \mathcal{K}$, we define the *norm* of A as

$$\|A\| := \sup\{\|x\| : x \in A\} = \sup_{\|p\|=1} spt(p, A).$$

where $\sup$ denotes the supremum. This definition satisfies all of the classical properties of a norm. The Hausdorff distance between $A, B \in \mathcal{K}$ can also be given in terms of the support function:

$$d_H(A, B) = \sup_{\|p\|=1} |spt(p, A) - spt(p, B)|.$$

Using this it is easy to show that addition and scalar multiplication are continuous in the Hausdorff distance.

We say that a set $A \subset \mathbb{R}^d$ is *balanced* if $\lambda A \subseteq A$ for all $|\lambda| \leq 1$. Any balanced $\mathbb{B} \in \mathcal{K}$ with $0 \in \text{int}(\mathbb{B})$ is a *unit ball* and any unit ball defines a norm on $\mathbb{R}^d$ via the Minkowski functional

$$\|x\|_{\mathbb{B}} = \sup\{\lambda \geq 0 : \lambda x \in \mathbb{B}\}.$$

The *dual sphere* is defined as

$$\mathbb{S}^* = \{y : \sup\{y \cdot x : x \in \mathbb{B}\} = 1\} \subset \mathbb{R}^d$$

and is a nonempty compact set. Notice that since $\mathbb{B}$ is compact, for each $y \in \mathbb{S}^*$, there is some $x \in \mathbb{B}$ with $y \cdot x = 1$. For more details see [21].

A *set-valued function* (or *multifunction*) taking compact and convex values is a map $f : \mathbb{R}^d \rightarrow \mathcal{K}$. For a given set-valued function f and measure μ on $\mathbb{R}^d$, we can define the integral of f with respect to μ via support functions using the property (see [4])

$$spt\left(q, \int_{\mathbb{R}^n} f(x) d\mu(x)\right) = \int_{\mathbb{R}^n} spt(q, f(x)) d\mu(x), \quad (2)$$

which defines the set as in (1). For more results on set-valued analysis see [4].

For a given a compact subset Θ of $\mathbb{R}^d$ and a set-valued function $f : \Theta \subseteq \mathbb{R}^d \rightarrow \mathcal{K}$, consider the optimization problem

$$\max_{x \in \Theta} f(x). \quad (3)$$

We say that $x_0 \in \Theta$ is a *global maximizer for f over Θ* if for any $x \in \Theta$ we have $f(x) \subseteq f(x_0)$. Notice that we are using the natural ordering of sets given by inclusion. A set-valued function $f : \mathbb{R} \rightarrow \mathcal{K}$ is *non decreasing* if $f(x) \subseteq f(y)$ whenever $x \leq y$ and is *convex* if

$$f(tx + (1-t)y) \subseteq tf(x) + (1-t)f(y) \quad (4)$$

for $x, y \in \mathbb{R}^d$, $t \in [0, 1]$.

Note that f is convex if and only if the function $spt(p, f(x))$ is convex for all $\|p\| = 1$. Further, f is *quasi-convex* if for any $x, y \in \mathbb{R}^d$ with $f(x) \subseteq f(y)$, we have that

$$f(tx + (1 - t)y) \subseteq f(y) \quad (5)$$

for all $t \in [0, 1]$.

3 Set-Valued Probabilities

Now we give a short overview of those aspects of set-valued measures and set-valued probabilities that we will need; for more information and proofs see [1, 2, 4, 12, 23, 24].

The idea of using set-valued probabilities is to extend the classical notion of probability where instead of the uncertainty about the occurrence of an event being expressed by a positive number we use a set. This can be very useful to model situations in which there is almost total ignorance and little or conflicting information about the system or subject under study. The classical axioms of probability theory are extended by means of the Minkowski sum and the inclusion order.

The notion of set-valued probability also extends the notion of imprecise probability [25–27] to the case of probabilities taking convex and compact-valued images.

Imprecise probabilities focus on describing uncertainty using only the best-case and worst-case scenarios. They use intervals to describe uncertainty about probabilities, focusing on the minimum and maximum likelihood of an event. Set-valued probabilities, instead, take this concept further by allowing probabilities to be expressed as more complex sets, enabling the representation of a wider range of uncertain scenarios. Set-valued probabilities allow for a richer and more nuanced representation of uncertainty, capturing complex scenarios beyond just the extremes.

Within the imprecise probability formulation a single probability specification ϕ is replaced with an interval specification by means of lower and upper probabilities, namely ϕ_- and ϕ_+ , and for a given event A we have that $\phi(A)$ is replaced by the positive interval $[\phi_-(A), \phi_+(A)] \subset [0, 1]$. The definition of imprecise probability can be recast in the framework of set-valued probability theory by assuming that the probability of a given event A is the interval $[-\phi_-(A), \phi_+(A)] \subset [-1, 1]$.

Starting with a given set Ω and a σ -algebra $\mathcal{A}$ on Ω , a *probability measure* on $(\Omega, \mathcal{A})$ is a function $\mu : \mathcal{A} \rightarrow [0, 1]$ such that $\mu(\emptyset) = 0$, $\mu(\Omega) = 1$, and

$$\mu\left(\bigcup_i A_i\right) = \sum_i \mu(A_i)$$

for any pairwise-disjoint sequence $A_i \in \mathcal{A}$. In a similar way, a *vector-valued probability measure* on $(\Omega, \mathcal{A})$ is a function $\mu : \mathcal{A} \rightarrow [0, 1]^d$, with $\mu(\emptyset) = (0, \dots, 0) \in \mathbb{R}^d$, $\mu(\Omega) = (1, \dots, 1) \in \mathbb{R}^d$, and

$$\mu\left(\bigcup_i A_i\right) = \sum_i \mu(A_i).$$

Note that here additivity is satisfied componentwise and each component is a probability measure.

The most general extension we consider is a *set-valued measure* or *multimeasure* with values in $\mathcal{K}$. This is a function $\Phi : \mathcal{A} \rightarrow \mathcal{K}$ such that $\Phi(\emptyset) = \{0\}$ and

$$\Phi\left(\bigcup_i A_i\right) = \sum_i \Phi(A_i). \quad (6)$$

The right side of (6) is the infinite Minkowski sum defined as

$$\sum_i K_i = \left\{ \sum_i k_i : k_i \in K_i, \sum_i |k_i| < \infty \right\},$$

which can also be defined as the Hausdorff limit of the partial sums.

Our assumption that $\Phi(A)$ is convex is not much of a restriction since this is always true for any bounded, non-atomic set-valued measure (see [1, 12]).

As with sets in $\mathcal{K}$, a multimeasure Φ is *nonnegative*, written $\Phi \geq 0$, if $0 \in \Phi(A)$ for all A . Nonnegative multimeasures are monotone: if $A \subseteq B$ then

$$\Phi(A) = \{0\} + \Phi(A) \subseteq \Phi(B \setminus A) + \Phi(A) = \Phi(B).$$

This makes nonnegative multimeasures a nice generalization of (nonnegative) scalar measures, which are also monotone.

For a multimeasure Φ and $p \in \mathbb{R}^d$, the *scalarization* Φ^p defined by

$$\Phi^p(A) = \text{spt}(p, \Phi(A)) \quad (7)$$

is easily seen to be a signed measure and a (positive) measure if $\Phi \geq 0$. To see this note that $0 \in \Phi(A)$ means $0 \leq \text{spt}(p, \Phi(A)) = \Phi^p(A)$ for any p .

Definition 1 Let $\mathbb{B} \in \mathcal{K}$ be a unit ball. A $\mathbb{B}$ *set-valued probability* or *probability multimeasure* (or pmm) on $(\Omega, \mathcal{A})$ is a nonnegative multimeasure Φ with $\Phi(\Omega) = \mathbb{B}$.

If Φ is a pmm, then for all $p \in \mathbb{S}$ we have that $\Phi^p \geq 0$, is monotone in A , and $\Phi^p(\Omega) = \text{spt}(p, \mathbb{B}) = 1$ and so Φ^p is a probability measure. Note that Φ^p and Φ^q are not “independent” but may be related in a complex manner. The primary constraint is that the mapping $p \mapsto \Phi^p(A)$ is convex for any $A \in \mathcal{A}$.

Just like for classical probability distributions, one simple and useful way to construct a multimeasure is by integrating a *set-valued density function* f with respect to a nonnegative scalar measure μ :

$$\Phi(A) = \int_A f(x) \, d\mu(x). \quad (8)$$

For us the most useful way of thinking of this integral is by the property given in (2), but there are other ways (see [4]). If the set-valued function f is nonnegative (that

is, $0 \in f(x)$ for all x), then the resulting multimeasure will also be nonnegative. In addition, if $0 \leq f(x) \leq g(x)$ and Φ is a nonnegative multimeasure, then

$$\int f(x) d\Phi(x) \subseteq \int g(x) d\Phi(x),$$

for this to be true the convexity of the values of Φ is crucial.

A *random variable* on $(\Omega, \mathcal{A})$ is simply a Borel measurable function $X : \Omega \rightarrow \mathbb{R}$. The expectation of X with respect to a pmm Φ is defined in the usual way as

$$\mathbb{E}_\Phi(X) := \int_\Omega X(\omega) d\Phi(\omega). \quad (9)$$

The integral can be constructed using support functions (that is, using the Φ^p), but a bit of care has to be taken if X has both positive and negative values, since subtraction in $\mathcal{K}$ does not work as we might wish. In the case that X has both positive and negative values, we use each part of the decomposition $X = X^+ - X^-$ separately. Since $0 \in \Phi(A)$ for each A , it is easy to see that $0 \in \mathbb{E}_\Phi(X)$ as well.

Classical results like the strong law of Large Numbers, the Glivenko-Cantelli, and the Central Limit Theorems can be extended to pmm (see [17] for more on this).

4 The Notion of Coherent Risk Measure

Given a random variable X , a coherent risk measure, $\rho(X)$, maps this random variable to a real number, which indicates the “riskiness” of X . A risk measure $\rho : \mathcal{X} \rightarrow \mathbb{R}$ is said to be **coherent** if it satisfies the following four properties:

- *Monotonicity*: If $X \leq Y$ almost surely, then $\rho(X) \geq \rho(Y)$. This property ensures that a position with lower returns is assessed as riskier.
- *Sub-additivity*: For all $X, Y \in \mathcal{X}$, $\rho(X + Y) \leq \rho(X) + \rho(Y)$. This reflects the idea of diversification: the risk of a combined portfolio should not exceed the sum of the individual risks.
- *Positive Homogeneity*: For $\lambda \geq 0$, $\rho(\lambda X) = \lambda \rho(X)$. This means that scaling the size of a position scales its risk linearly.
- *Translation Invariance*: For all $a \in \mathbb{R}$, $\rho(X + a) = \rho(X) - a$. Adding a risk-free cash amount a to a position reduces the risk by a .

A coherent risk measure $\rho(X)$ for a (random variable) X can be represented in its dual form as follows ([3, 14]):

$$\rho(X) = \sup_{\mu \in \mathcal{Q}} \mathbb{E}_\mu[-X],$$

where $\mathcal{Q}$ is a set of probability measures (often referred to as a set of “dual measures”), $\mathbb{E}_\mu[-X]$ is the expectation of the negative of the position X under the probability measure μ . Therefore, the risk measure is the worst-case expectation over all measures in $\mathcal{Q}$.

The dual representation connects coherent risk measures to robust optimization and provides the following intuitive interpretations: The set $\mathcal{Q}$ can be thought of as a family of “adverse” or “stress” probability measures that capture uncertainty in the modeling of the position X while the risk measure $\rho(X)$ is the maximum expected loss under these adverse measures.

Let us now consider a portfolio (not necessarily including financial assets) consisting of n assets, where the return of the i -th asset is denoted by X_i . The portfolio return X is expressed as a linear combination of the individual asset returns:

$$X = \sum_{i=1}^n \alpha_i X_i,$$

where α_i represents the weight assigned to the i -th asset in the portfolio, and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is the vector of portfolio weights. The goal is to minimize the risk of the portfolio while satisfying certain constraints. If the risk is measured using a coherent risk measure $\rho(X)$, the optimization problem can be formulated as:

$$\begin{aligned} \min_{\alpha} \quad & \rho \left(\sum_{i=1}^n \alpha_i X_i \right), \\ \text{subject to} \quad & \sum_{i=1}^n \alpha_i = 1, \quad (\text{budget constraint}) \\ & \alpha_i \geq 0 \quad \forall i, \\ & \alpha \in \mathbb{R}^n. \end{aligned}$$

where $\rho(X)$ is a coherent risk measure (for instance Conditional Value at Risk (CVaR) or another risk metric), $\sum_{i=1}^n \alpha_i = 1$ ensures that the total portfolio weights sum to 1, reflecting full investment of available capital, and $\alpha_i \geq 0$ ensures that all weights are non-negative.

5 Φ -Coherent Risk Measure

In this subsection, we introduce a Φ -coherent variant of the classical coherent risk measure. Let us consider a probability space $(\Omega, \mathcal{F}, \mu)$, where Ω is the sample space, $\mathcal{F}$ is the σ -algebra of events, and $\mathcal{Q}$ is a set of potential probability measures. For a random variable $X : \Omega \rightarrow \mathbb{R}$, a coherent risk measure $\rho(X)$ is defined as:

$$\rho(X) = \sup_{\mu \in \mathcal{Q}} \mathbb{E}_{\mu}[-X] \tag{10}$$

where $\mathbb{E}_{\mu}$ denotes the expectation with respect to the probability measure μ . We now extend this classical framework by incorporating the notion of set-valued probability Φ .

Definition 2 Given a positive random variable X and a set-valued probability Φ taking values in $\mathbb{R}^d$, a ρ_{Φ} coherent risk measure is defined as

$$\rho_{\Phi}(X) = \mathbb{E}_{\Phi}[-X] \quad (11)$$

The following result presents some properties of the risk measure ρ_{Φ} which are analogous to well-known properties of single-valued coherent risk measures.

Theorem 1 *Given two positive random variables X and Y , a constant c , and a positive real number t , the risk measure ρ_{Φ} satisfies the following properties;*

- $\rho_{\Phi}(c) = c\mathbb{B}$ where $\mathbb{B}$ is the unit ball in $\mathbb{R}^d$
- $\rho_{\Phi}(X + Y) = \rho_{\Phi}(X) + \rho_{\Phi}(Y)$
- $\rho_{\Phi}(tX) = t\rho_{\Phi}(X)$
- $\rho_{\Phi}(tX + (1-t)Y) = t\rho_{\Phi}(X) + (1-t)\rho_{\Phi}(Y)$ for $0 \leq t \leq 1$
- If $X \leq Y \leq c$, where c is positive constant, then $\rho_{\Phi}(c - Y) \subseteq \rho_{\Phi}(c - X)$

Proof The following proofs are quite straightforward.

- We have that $\rho_{\Phi}(c) = \mathbb{E}_{\Phi}[-c]d\Phi = -c\Phi(\Omega) = -c\mathbb{B} = c\mathbb{B}$
- Computing, we have:

$$\rho_{\Phi}(X + Y) = \mathbb{E}_{\Phi}[-(X + Y)] = \int_{\Omega} (-X - Y)d\Phi =$$

$$\int_{\Omega} -Xd\Phi + \int_{\Omega} -Yd\Phi = \rho_{\Phi}(X) + \rho_{\Phi}(Y)$$

- We have:

$$\rho_{\Phi}(tX) = \mathbb{E}_{\Phi}[-tX] = t\mathbb{E}_{\Phi}(-X) = t\rho_{\Phi}(X)$$

- It follows from the first two properties
- If $X \leq Y \leq c$ where c is a positive constant, then $0 \leq c - Y \leq c - X$ and for any $p \in S^*$, we have

$$spt(p, \rho_{\Phi}(Y - c)) = spt(p, \mathbb{E}_{\Phi}[c - Y]) = \int_{\Omega} (c - Y)d\Phi^p(\omega) \leq$$

$$\int_{\Omega} (c - X)d\Phi^p(\omega) = spt(p, \rho_{\Phi}(c - X))$$

and this implies that $\rho_{\Phi}(c - Y) \subseteq \rho_{\Phi}(c - X)$.

□

6 Risk Minimization using a Φ -Coherent Risk Measure

In sustainability, coherent risk measures can quantify the risk of exceeding critical environmental thresholds. Let X_i represent the environmental impact of a project

or activity, such as greenhouse gas emissions, water consumption, or waste generation. A Φ -robust coherent risk measure $\rho_\Phi(X_i)$ can be used to assess and minimize sustainability risks.

As in the previous part of this paper, let us now consider a portfolio consisting of n assets, where the return of the i -th asset is denoted by X_i . The portfolio return X is expressed as a linear combination of the individual asset returns:

$$X = \sum_{i=1}^n \alpha_i X_i,$$

where α_i represents the weight assigned to the i -th asset in the portfolio, and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is the vector of portfolio weights. The goal is to minimize the risk of the portfolio which is measured by means of a Φ coherent risk measure ρ_Φ :

$$\rho_\Phi \left(\sum_{i=1}^n \alpha_i X_i \right) = \mathbb{E}_\Phi \left[- \sum_{i=1}^n \alpha_i X_i \right] = \sum_{i=1}^n \alpha_i \mathbb{E}_\Phi [-X_i] \quad (12)$$

Therefore, the risk minimization problem reads as

$$\min_{\alpha_1, \dots, \alpha_n} \sum_{i=1}^n \alpha_i \mathbb{E}_\Phi [-X_i] \quad (13)$$

subject to the following constraints:

$$\begin{cases} \sum_{i=1}^n \alpha_i = 1, & \text{(resource allocation constraint)} \\ \alpha_i \geq 0 \quad \forall i, & \text{(non-negativity constraint).} \end{cases}$$

The following definition introduces the notion of minimizer for the above optimization model.

Definition 3 A vector $(\alpha_1^*, \dots, \alpha_n^*)$ is said to be an optimal ideal solution for the risk minimization problem if the following inclusion holds:

$$\sum_{i=1}^n \alpha_i^* \mathbb{E}_\Phi [-X_i] \subseteq \sum_{i=1}^n \alpha_i \mathbb{E}_\Phi [-X_i] \quad (14)$$

for any feasible vector $(\alpha_1, \dots, \alpha_n)$.

Definition 4 A vector $(\alpha_1^*, \dots, \alpha_n^*)$ is said to be a strong optimal solution for the risk minimization problem if there exists no feasible vector $(\alpha_1, \dots, \alpha_n)$ such that

$$\sum_{i=1}^n \alpha_i \mathbb{E}_\Phi [-X_i] \subset \sum_{i=1}^n \alpha_i^* \mathbb{E}_\Phi [-X_i]. \quad (15)$$

Definition 5 A vector $(\alpha_1^*, \dots, \alpha_n^*)$ is said to be an optimal solution for the risk minimization problem if there exists no feasible vector $(\alpha_1, \dots, \alpha_n)$ such that

$$\sum_{i=1}^n \alpha_i \mathbb{E}_\Phi [-X_i] \subset \text{int} \sum_{i=1}^n \alpha_i^* \mathbb{E}_\Phi [-X_i]. \quad (16)$$

The following result states a scalarization property for the risk minimization problem.

Theorem 2 A feasible vector $(\alpha_1^*, \dots, \alpha_n^*)$ is an ideal optimal solution if and only if it is an optimal solution for the problem

$$\min_{\alpha_1, \dots, \alpha_n} \sum_{i=1}^n \alpha_i \text{spt}(p, \mathbb{E}_\Phi [-X_i]) \quad (17)$$

subject to:

$$\begin{cases} \sum_{i=1}^n \alpha_i = 1, \\ \alpha_i \geq 0, \quad \forall i \end{cases}$$

for any direction $p \in \mathbb{S}$. Let us notice that the scalarized problem is a classical linear optimization model.

Proof The proof is based on the fact that showing

$$\sum_{i=1}^n \alpha_i^* \mathbb{E}_\Phi [-X_i] \subseteq \sum_{i=1}^n \alpha_i \mathbb{E}_\Phi [-X_i] \quad (18)$$

for any feasible vector $(\alpha_1, \dots, \alpha_n)$ is equivalent to proving that

$$\sum_{i=1}^n \alpha_i^* \text{spt}(p, \mathbb{E}_\Phi [-X_i]) \leq \sum_{i=1}^n \alpha_i \text{spt}(p, \mathbb{E}_\Phi [-X_i]) \quad (19)$$

for any direction $p \in \mathbb{S}$. This follows easily from the properties of the support function and the fact that each $\mathbb{E}_\Phi [-X_i]$ is a compact and convex set in $\mathbb{R}^d$. $\square$

Corollary 6.1 Let us suppose that X_i are positive random variables bounded above by a constant $c > 0$. Then

$$\text{spt}(p, \mathbb{E}_\Phi [c - X_i]) = \text{spt}(p, \mathbb{E}_\Phi [c]) + \text{spt}(p, \mathbb{E}_\Phi [-X_i]) = c + \text{spt}(p, \mathbb{E}_\Phi [-X_i]),$$

and thus the above optimization model is equivalent to:

$$\min_{\alpha_1, \dots, \alpha_n} \sum_{i=1}^n \alpha_i \mathbb{E}_{\Phi^p} [c - X_i] = \sum_{i=1}^n \alpha_i \mathbb{E}_{\Phi^p} [-X_i] \quad (20)$$

subject to:

$$\begin{cases} \sum_{i=1}^n \alpha_i = 1, \\ \alpha_i \geq 0, \quad \forall i. \end{cases}$$

Proof The proof follows by applying previous results to the positive random variable $c - X_i$. $\square$

Theorem 3 Let $\alpha = (\alpha_1, \dots, \alpha_n)$ and for a feasible vector $\alpha^* = (\alpha_1^*, \dots, \alpha_n^*)$ consider the function

$$g_{\alpha^*}(\alpha) = \sup_{p \in \mathbb{S}^*} \left(\sum_{i=1}^n \alpha_i \text{spt}(p, \mathbb{E}_{\Phi}[-X_i]) - \sum_{i=1}^n \alpha_i^* \text{spt}(p, \mathbb{E}_{\Phi}[-X_i]) \right).$$

Then if α^* is a strong minimizer then α^* is also a solution to

$$\min_{\alpha} g_{\alpha^*}(\alpha). \quad (21)$$

Conversely, if α^* is a strict minimizer for (21), then α^* is a strong minimizer. If α^* is a minimizer for problem (21), then it is a minimizer.

Proof Assume α^* is a strong minimizer. Hence, for every feasible α we have

$$\sum_{i=1}^n \alpha_i \mathbb{E}_{\Phi}[-X_i] \not\leq \sum_{i=1}^n \alpha_i^* \mathbb{E}_{\Phi}[-X_i]$$

This implies that for every feasible α there exists $p \in \mathbb{S}^*$ such that

$$\text{spt}(p, \sum_{i=1}^n \alpha_i \mathbb{E}_{\Phi}[-X_i]) \geq \text{spt}(p, \sum_{i=1}^n \alpha_i^* \mathbb{E}_{\Phi}[-X_i])$$

This entails

$$\sup_{p \in \mathbb{S}^*} \left(\text{spt}(p, \sum_{i=1}^n \alpha_i \mathbb{E}_{\Phi}[-X_i]) - \text{spt}(p, \sum_{i=1}^n \alpha_i^* \mathbb{E}_{\Phi}[-X_i]) \right) \geq 0$$

which means α^* is minimal for $g_{\alpha^*}(\alpha)$, since $g_{\alpha^*}(\alpha^*) = 0$.

Assume now that $\alpha^* \in X$ is a strict minimum for $g_{\alpha^*}(\alpha)$, that is that $g_{\alpha^*}(\alpha) > g_{\alpha^*}(\alpha^*) = 0$ for all feasible α . Then for any $\alpha \neq \alpha^*$ we have that there is some $p \in \mathbb{S}^*$ with

$$\text{spt}(p, \sum_{i=1}^n \alpha_i^* \mathbb{E}_{\Phi}[-X_i]) < \text{spt}(p, \sum_{i=1}^n \alpha_i \mathbb{E}_{\Phi}[-X_i])$$

which means that

$$\sum_{i=1}^n \alpha_i \mathbb{E}_{\Phi}[-X_i] \not\leq \sum_{i=1}^n \alpha_i^* \mathbb{E}_{\Phi}[-X_i]$$

and so α^* is a strong minimizer as claimed.

Next assume α^* is a minimum of $g_{\alpha^*}(\alpha)$. For every α we have the existence of $p \in S^*$ such that

$$\text{spt}(p, \sum_{i=1}^n \alpha_i^* \mathbb{E}_{\Phi}[-X_i]) \leq \text{spt}(p, \sum_{i=1}^n \alpha_i \mathbb{E}_{\Phi}[-X_i])$$

This means that we cannot have

$$\text{spt}(p, \sum_{i=1}^n \alpha_i^* \mathbb{E}_{\Phi}[-X_i]) > \text{spt}(p, \sum_{i=1}^n \alpha_i \mathbb{E}_{\Phi}[-X_i])$$

for every $p \in S^*$, which is equivalent to

$$\sum_{i=1}^n \alpha_i \mathbb{E}_{\Phi}[-X_i] \not\leq \text{int} \sum_{i=1}^n \alpha_i^* \mathbb{E}_{\Phi}[-X_i]$$

and the proof is complete. $\square$

Corollary 6.2 If for some $p^* \in \mathbb{S}^*$, α^* is a strict minimizer for

$$\min_{\alpha} spt(p^*, \sum_{i=1}^n \alpha_i \mathbb{E}_{\Phi}[-X_i])$$

i.e.

$$spt(p^*, \sum_{i=1}^n \alpha_i^* \mathbb{E}_{\Phi}[-X_i]) < spt(p^*, \sum_{i=1}^n \alpha_i \mathbb{E}_{\Phi}[-X_i])$$

for every feasible $\alpha \neq \alpha^*$, then α^* is a strong minimizer.

Proof This follows directly from the argument given in the previous theorem. $\square$

Now we give a stability condition.

Theorem 4 Suppose that for each $i = 1, 2, \dots, n$, we have that $X_{i,m} \rightarrow X_i$ uniformly on Ω where X_i is bounded and Lipschitz on Ω . Let Φ_m be a sequence of set-valued probabilities that are converging to Φ in the extended Monge-Kantorovich metric, and let α_m be a sequence of feasible weights converging to a feasible α^* . If α_m is an ideal minimizer for the problem

$$\min_{\beta} \sum_{i=1}^n \beta_i \mathbb{E}_{\Phi_m}[-X_{i,m}]$$

then α^* is an ideal minimizer for the problem

$$\min_{\beta} \sum_{i=1}^n \beta_i \mathbb{E}_{\Phi}[-X_i]. \quad (22)$$

Proof We see that for any fixed i and $p \in \mathbb{S}$ and any arbitrary sequence $a_m \in \mathbb{R}_+$ with $a_m \rightarrow a$, $a \in \mathbb{R}_+$, for large enough m we have

$$\begin{aligned} |spt(p, a_m \mathbb{E}_{\Phi_m}[-X_{i,m}]) - spt(p, a \mathbb{E}_{\Phi}[-X_i])| &\leq \\ &a_m |spt(p, \mathbb{E}_{\Phi_m}[-X_{i,m}]) - spt(p, \mathbb{E}_{\Phi_m}[-X_i])| \\ &+ a_m |spt(p, \mathbb{E}_{\Phi_m}[-X_i]) - spt(p, \mathbb{E}_{\Phi}[-X_i])| \\ &+ |a_m - a| |spt(p, \mathbb{E}_{\Phi}[-X_i])| \\ &\leq 2a \|\mathbb{B}\| \|X_{i,m} - X_i\|_{\infty} \\ &+ 2ad_{MK}(\Phi_m, \Phi) \\ &+ |a_m - a| \|\mathbb{E}_{\Phi}[-X_i]\|. \end{aligned}$$

Since each of these terms can be made arbitrarily small, we have that

$$spt(p, a_m \mathbb{E}_{\Phi_m}[-X_{i,m}]) \rightarrow spt(p, a \mathbb{E}_{\Phi}[-X_i]).$$

This means that

$$spt(p, \sum_i (\alpha_m)_i \mathbb{E}_{\Phi_m}[-X_{i,m}]) \rightarrow spt(p, \sum_i \alpha_i^* \mathbb{E}_{\Phi}[-X_i])$$

for any $p \in \mathbb{S}$ and thus the result follows. $\square$

7 An Illustrative Example

In the face of climate change, environmental sustainability projects have become increasingly important. However, these projects often involve risks, such as unintended ecological impacts or inefficiencies. This case study evaluates five green initiatives and formulates a mathematical optimization model to minimize the total environmental risk while ensuring non-overlapping resource use and impact.

The five activities considered are **tree planting**, **solar panel installations**, **wind energy production**, **urban gardens**, and **recycling programs**. These activities are associated with risk intervals representing uncertainties in their environmental impact. The goal is to select a subset of activities with the lowest total environmental risk.

Activity	Description	$\mathbb{E}_{\Phi}[-X_i]$
A	Large-scale tree planting to absorb CO ₂ emissions	[-0.4, -0.3]
B	Installing solar panels to reduce reliance on fossil fuels	[-0.7, -0.2]
C	Building wind turbines for renewable energy production	[-0.6, -0.4]
D	Developing urban green spaces to combat heat islands	[-0.5, -0.1]
E	Launching large-scale recycling initiatives	[-0.8, -0.5]

The risk minimization problem can be written as:

$$\min_{\alpha_1, \dots, \alpha_5} \alpha_1[-0.4, -0.3] + \alpha_2[-0.7, -0.2] + \alpha_3[-0.6, -0.4] + \alpha_4[-0.5, -0.1] + \alpha_5[-0.8, -0.5] =$$

$$\min_{\alpha_1, \dots, \alpha_5} [-0.4\alpha_1 - 0.7\alpha_2 - 0.6\alpha_3 - 0.5\alpha_4 - 0.8\alpha_5, -0.3\alpha_1 - 0.2\alpha_2 - 0.4\alpha_3 - 0.1\alpha_4 - 0.5\alpha_5]$$

subject to the following constraints:

$$\begin{cases} \sum_{i=1}^5 \alpha_i = 1, & \text{(resource allocation constraint)} \\ \alpha_i \geq 0 \quad \forall i, & \text{(non-negativity constraint).} \end{cases}$$

To determine an ideal optimal solution, we need to find the smallest interval that is included in all others. The above model can be decomposed into two different scalar models:

$$\min_{\alpha_1, \dots, \alpha_5} -0.3\alpha_1 - 0.2\alpha_2 - 0.4\alpha_3 - 0.1\alpha_4 - 0.5\alpha_5 \Rightarrow \text{minimize the best-case scenario}$$

and

$$\max_{\alpha_1, \dots, \alpha_5} -0.4\alpha_1 - 0.7\alpha_2 - 0.6\alpha_3 - 0.5\alpha_4 - 0.8\alpha_5 \Rightarrow \text{maximize the worst-case scenario}$$

subject to the above constraints. The model can be solved (approximatively) by means of linear scalarization techniques with weights w_1 and w_2 , $w_1 + w_2 = 1$, as follows:

$$\min_{\alpha_1, \dots, \alpha_5} (-0.3w_1 + 0.4w_2)\alpha_1 + (-0.2w_1 + 0.7w_2)\alpha_2 + (-0.4w_1 + 0.6w_2)\alpha_3 + \quad (23)$$

$$(-0.1w_1 + 0.5w_2)\alpha_4 + (-0.5w_1 + 0.8w_2)\alpha_5$$

where the previous maximization problem has been transformed into a minimization problem before being combined together. The following Table 1 shows the results.

w_1	w_2	$\alpha_1, \dots, \alpha_5$
0.0	1.0	[1, 0, 0, 0, 0]
0.1	0.9	[1, 0, 0, 0, 0]
0.2	0.8	[1, 0, 0, 0, 0]
0.3	0.7	[1, 0, 0, 0, 0]
0.4	0.6	[1, 0, 0, 0, 0]
0.5	0.5	[1, 0, 0, 0, 0]
0.6	0.4	[1, 0, 0, 0, 0]
0.7	0.3	[0, 0, 0, 0, 1]
0.8	0.2	[0, 0, 0, 0, 1]
0.9	0.1	[0, 0, 0, 0, 1]
1.0	0.0	[0, 0, 0, 0, 1]

Table 1 Results for different weight combinations (w_1, w_2) .

The results provide values for α_i (where $i = 1, 2, \dots, 5$) for different weight combinations $(w_1$ and $w_2)$. These values represent how much of the resources should be allocated to each intervention based on the trade-off between (w_1) and (w_2) . Each α_i corresponds to an intervention:

- α_1 : Tree planting to absorb CO₂.
- α_2 : Solar panels to reduce reliance on fossil fuels.
- α_3 : Wind turbines for renewable energy.
- α_4 : Urban green spaces to combat heat islands.
- α_5 : Recycling initiatives on a large scale.

The optimal allocation changes depending on how much weight $(w_1$ or $w_2)$ is placed on the objectives:

- If you prioritize *minimizing the best-case outcomes* ($w_1 = 1$), all resources should go to tree planting to absorb CO₂, as they reduce the best-case impact the most.
- If you prioritize *maximizing the worst-case outcomes* ($w_2 = 1$), all resources should go to recycling initiatives on a large scale.
- For other weights, the solution always allocates resources to a single intervention, suggesting that the interventions are independent in effectiveness.

Remark 1 The previous approach can be generalized to incorporate 2D set-valued measures, allowing for the inclusion of additional risks linked to various economic factors, among others. In this extended framework, the expected value of each activity is represented as a set in $\mathbb{R}^2$ rather than in $\mathbb{R}$. This two-dimensional setting encapsulates multiple criteria or dimensions that are relevant to the decision-making process. For instance, for two-dimensional set-valued probability measures, the table would read, for instance, as follows:

Activity	Description	$\mathbb{E}_{\Phi}[-X_i]$
A	Large-scale tree planting	$[-0.4, -0.3] \times [-0.5, -0.2]$
B	Installing solar panels	$[-0.7, -0.2] \times [-0.5, -0.1]$
C	Building wind turbines	$[-0.6, -0.4] \times [-0.5, -0.1]$
D	Developing urban green spaces	$[-0.5, -0.1] \times [-0.5, -0.4]$
E	Launching large-scale recycling initiatives	$[-0.8, -0.5] \times [-0.6, -0.2]$

To identify the optimal solution within this framework, the model can be scalarized, similar to the method outlined in the previous case. Scalarization involves reducing the multi-dimensional problem into a single-dimensional one by applying weights to the different criteria. These weights reflect the relative importance or priority assigned to each criterion, essentially combining them into a single objective function. Once scalarized, the problem can be formulated as a linear programming model. Different sets of scalarization weights can be employed to explore trade-offs among the criteria and to accommodate varying preferences or priorities. For brevity, we do not present the details of this process here, as it is quite similar to the example previously discussed and the same principles and techniques would apply, making the extension straightforward to implement.

8 Conclusion

Coherent risk measures provide a powerful and mathematically rigorous framework for quantifying risk, making them highly valuable in sustainability applications. By incorporating the principles of monotonicity, sub-additivity, positive homogeneity, and translation invariance, coherent risk measures ensure consistency and robustness in risk assessments. In the context of sustainability, where risks are uncertain, multidimensional, and often extreme, the need for robust and reliable definitions is particularly acute. The use of set-valued probabilities offers a promising extension of coherent risk measures, enabling them to handle uncertainty in probability distributions and multidimensional risk scenarios. This approach enhances the applicability and reliability of coherent risk measures, making them indispensable tools for tackling the complex and urgent challenges of sustainability. We have provided an illustrative example in the area of risk minimization. Our case study evaluated five green initiatives—tree planting, solar panels, wind turbines, urban green spaces, and recycling programs. The analysis revealed that the optimal allocation of resources depends on how worst and best scenario risks are prioritized. If minimizing the best-case outcomes is prioritized, then all resources should go to tree planting to absorb CO₂, as they reduce the best-case impact the most. If, instead, maximizing the worst-case outcomes is prioritized, then all resources should go to recycling initiatives on a large scale. For other weights, the solution always allocates resources to a single intervention, suggesting that the interventions are independent in effectiveness. Finally, the adaptability

of ϕ -coherent risk measures extends far beyond their use in sustainability. These measures offer a highly flexible framework for addressing diverse and complex risks, thanks to their ability to handle uncertainties, trade-offs, and multi-dimensional challenges effectively in a variety of different domains such as supply chain management, healthcare systems, engineering design, disaster risk management, and cybersecurity.

Compliance with Ethical Standards

The research of F. Mendivil is partially supported by NSERC 2019:05237.

Ethical approval

This article does not contain any studies with human participants performed by any of the authors.

References

- [1] Z. Artstein, Set-valued measures, *Transactions of the American Mathematical Society*, **165** (1972), 103-125.
- [2] Z. Artstein, On the calculus of closed set-valued functions, *Indiana Univeristy Mathematical Journal*, **24** (1974), 433-441.
- [3] Artzner, P., Delbaen, F., Eber, J. M., and Heath, D. (1999). Coherent measures of risk. *Mathematical finance*, 9(3), pp. 203–228.
- [4] J.P. Aubin, H. Frankowska, *Set-valued analysis*, Birkhäuser, Boston, 1990.
- [5] F. Ben Abdelaziz, H. Masri, Stochastic programming with fuzzy linear partial information on probability distribution, *European Journal of Operational Research*, **162**, (2005), no. 3, 619-629.
- [6] F. Ben Abdelaziz, H. Masri, A compromise solution for the multiobjective stochastic linear programming under partial uncertainty, *European Journal of Operational Research*, **202**, (2010), 55-59.
- [7] F. Ben Abdelaziz, D. La Torre, Robust generalized Merton-type financial portfolio models with generalized utility, *Annals of Operations Research*, (2021), in press. <https://doi.org/10.1007/s10479-021-04051-x>
- [8] L. A. Ben-Tal, L. El Ghaoui, A. Nemirovski, *Robust optimization*, Princeton University Press, USA, 2009.
- [9] A. P. Dempster (1966) New approaches for reasoning towards posterior distributions based on sample data. *Ann. Math. Statist.* 37 355–374.

- [10] D. Dubois, H. Prade (1988) *Possibility theory an approach to computerized processing of uncertainty*, Plenum Press, New York.
- [11] Y. Ermoliev, A. Gaivoronski, Stochastic optimization problems with incomplete information on distribution functions, *SIAM Journal on Control and Optimization*, **23**, (1985), no. 5, 697–716.
- [12] C. Hess, Set-valued integration and set-valued probability theory: an overview, in *Handbook of Measure Theory, vols 1, II*, North-Holland, Amsterdam (2002).
- [13] D. Kuroiwa, Existence theorems of set optimization with set-valued maps, *Journal of Information and Optimization sciences*, **24**, (2003), no. 1, 73–84.
- [14] Jouini, E., Meddeb, M., Touzi, N. (2004) Vector-valued coherent risk measures. *Finance Stochastics* , 8, pp. 531– 552
- [15] D. La Torre, F. Mendivil, The Monge–Kantorovich metric on multimeasures and self-similar multimeasures, *Set-Valued and Variational Analysis* **23** (2015), no. 2, 319–331.
- [16] D. La Torre, F. Mendivil, Portfolio optimization under partial uncertainty and incomplete information: a probability multimeasure-based approach, *Annals of Operations Research*, **267** (2018), 267–279.
- [17] D. La Torre, F. Mendivil, Stochastic linear optimization under partial uncertainty and incomplete information using the notion of probability multimeasure, *Journal of the Operational Research Society*, **69** (2018), no. 10, 1549–1556.
- [18] La Torre, D., Mendivil, F. Stochastic efficiency and inefficiency in portfolio optimization with incomplete information: a set-valued probability approach, *Annals of Operations Research*, (2020), in press. <https://doi.org/10.1007/s10479-020-03886-0>
- [19] H. Markowitz, Portfolio Selection, *The Journal of Finance* **7** (1952), 77–91.
- [20] F. Mendivil, Computing the Monge-Kantorovich distance, *Comp. Appl. Math.*, **36**(3) (2017), 1989–1402.
- [21] R.T. Rockafellar, R. J-B. Wets, *Variational analysis*, Springer, New York, 1998.
- [22] A. Shapiro, D. Dentcheva, and A. Ruszczyński. *Lectures on stochastic programming: modeling and theory*, SIAM, 2014.
- [23] M. Stojaković, Set valued probability and its connection with set valued measure, *Statist. Probab. Lett.* **82** (2012), 1043–1048.
- [24] K. Vind (1964), Edgeworth-allocations in an exchange economy with many traders, *Int. Econ. Rev.* 5(2), 165–177.

- [25] P. Walley (1991), *Statistical reasoning with imprecise probabilities*, Chapman and Hall, London, New York.
- [26] Y. Wang (2008), Imprecise probabilities with a generalized interval form. In R.L. Muhanna and R.L. Mullen, eds., Proc. 3rd Int. Workshop on Reliability Engineering Computing (REC'08), Savannah, Georgia, 45-59.
- [27] K. Weichselberger (2000), The theory of interval-probability as a unifying concept for uncertainty, *International Journal of Approximate Reasoning*, 24 (2–3), pp. 149-170.
- [28] United Nations. Trillion Trees Initiative. <https://www.trilliontrees.org>, accessed: 2023-10-01.
- [29] R. L. Chazdon and P. H. S. Brancalion. Restoring forests as a means to many ends. *Science*, 365(6448):24–25, 2019.
- [30] International Energy Agency (IEA). Solar energy. <https://www.iea.org/topics/renewables/solar>, accessed: 2023-10-01.
- [31] V. M. Fthenakis and H. C. Kim. Photovoltaics: Life-cycle analyses. *Renewable and Sustainable Energy Reviews*, 15(8):3266–3270, 2011.
- [32] National Renewable Energy Laboratory (NREL). Wind energy. <https://www.nrel.gov>, accessed: 2023-10-01.
- [33] S. R. Loss, T. Will, and P. P. Marra. Estimates of bird collision mortality at wind facilities in the contiguous United States. *Biological Conservation*, 168:201–209, 2013.
- [34] World Health Organization (WHO). Urban green spaces: A brief for action. <https://www.who.int/publications>, accessed: 2023-10-01.
- [35] V. Jennings and O. Bamkole. The relationship between social cohesion and urban green space: An avenue for health promotion. *International Journal of Environmental Research and Public Health*, 16(3):452, 2019.
- [36] United States Environmental Protection Agency (EPA). Recycling benefits report. <https://www.epa.gov/recycle>, accessed: 2023-10-01.
- [37] J. Hopewell, R. Dvorak, and E. Kosior. Plastics recycling: Challenges and opportunities. *Philosophical Transactions of the Royal Society B: Biological Sciences*, 364(1526):2115–2126, 2009.