Thin sets with fat shadows: projections of Cantor Sets

F. Mendivil, T. D. Taylor

1 INTRODUCTION

A Cantor set is a nonempty, compact, totally disconnected, perfect subset of $\mathbb{R}^n$. Now, the set being totally disconnected means that it is scattered about like a "dust". If you shine light on a clump of dust floating in the air, the shadow of this dust will look like a bunch of spots on the wall. You would be very surprised if you saw that the shadow was a filled-in shape (like a rabbit, say!). That would be pretty unbelievable. So, is this possible? We can think of the projection of a Cantor set onto a subspace as the shadow on that subspace. Is it possible that a cloud of dust (a Cantor set) could have a shadow (projection) which is "filled-in" (homeomorphic to the $n-1$ dimensional unit ball)? The answer is YES! In fact, it is possible to have the shadow in *every* direction be "filled-in"!

In this note we give an example of a simple construction of a Cantor subset of the unit square whose projection in every direction is a line segment. This construction can easily be generalized to n dimensions.

2 CONSTRUCTION

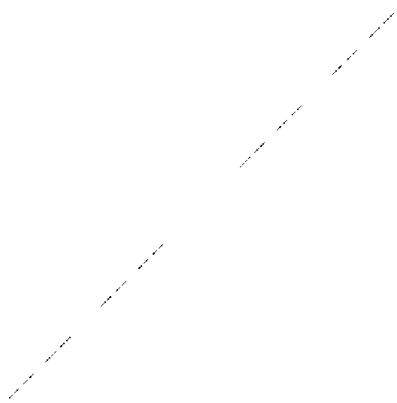

Figure 1: A simple Cantor set.

For the moment let's consider only Cantor subsets of the plane (in fact, the unit square), such as the one in Figure 1. For this particular set, if you project it any direction, you will always get a Cantor set (except for the one direction along the set for which you get a single point). Thus, none of the shadows for this set are "filled-in" (or contain intervals). It is worthwhile taking the time to convince yourself of this.

So how does one make a Cantor set which *will* project to a line segment, at least in one direction? One simple idea is illustrated in Figure 2. In this figure, we illustrate the deterministic generation of a Cantor set by starting with the unit square and successively transforming it. This sequence of sets will converge to the Cantor set shown on the lower right in Figure 2.

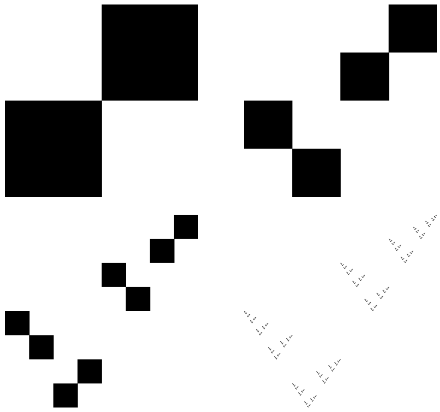

Figure 2: A dyadic "diagonal" Cantor set.

In this construction, we use the two maps given by

$$w_0(x, y) = (-x/2 + 1/2, y/2) \quad \text{and} \quad w_1(x, y) = (x/2 + 1/2, y/2 + 1/2).$$

The first panel (top left) in Figure 2 shows the images of the unit square under these two maps. The second panel (top right) shows the second step of the iteration where we apply the two maps to the set in the previous panel, and illustrates the horizontal "flip" caused by w_0.

Now, clearly the unit square S projects onto the line segment $[0, 1]$ both horizontally and vertically. Since the combined action of w_0 and w_1 preserves this property (as is obvious from Figure 2), $w_0(S) \cup w_1(S)$ also projects onto the line segment $[0, 1]$ both horizontally and vertically. This property is preserved for each subsequent iteration, and thus the limiting Cantor set also has this property.

This construction is an example of a general method for constructing fractal sets using an *Iterated Function System*, or IFS for short. An IFS on $I\!R^n$ is simply a collection of contractions from $I\!R^n$ into itself (such as the two maps w_0 and w_1, given above). We combine the action of the various maps into one set-valued map $\widehat{w}(B) = w_0(B) \cup w_1(B)$ and iterate to produce the *attractor* of the IFS. More formally, the attractor of an IFS $\{w_i\}$ is the unique nonempty compact set A which satisfies $A = w_0(A) \cup w_1(A) \cup \cdots \cup w_N(A)$. From the general theory of IFS's (see any of the references below), we know that any IFS has a unique attractor.

In Figure 3 we show four other Cantor subsets of the unit square which project onto the line segment $[0, 1]$ both horizontally and vertically. Each set is the unique attractor of an IFS, with the first IFS having two maps and the later three having three maps each.

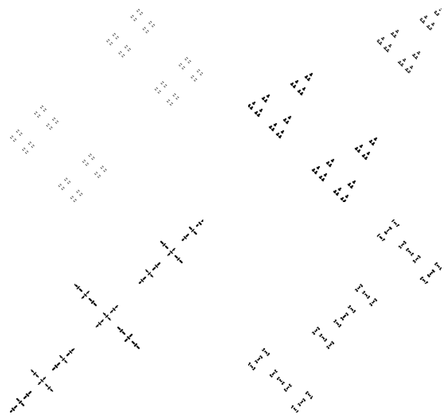

Figure 3: Four other projecting Cantor sets.

Now consider the diagonal line of the unit square with slope equal to 1. The projection of any of the four Cantor sets in Figure 3 onto this diagonal is disconnected (contains gaps). One way to see this is to consider lines of slope -1 and whether they intersect with the given Cantor set. Clearly, the only relevant lines are those which intersect the convex hull of the given Cantor set. By looking at the figure, we can see that many such lines will not intersect the given Cantor set, and these correspond to gaps in the projection. In fact, it is not hard to show that the projection is also a Cantor set.

At this point, we wish to try to construct an IFS whose attractor is a Cantor set which projects onto a line segment in every possible direction. One possible construction is illustrated in Figure 4. This Cantor subset of the unit square is the attractor of an IFS involving four contractions $\{w_0, w_1, w_2, w_3\}$, as illustrated in the first image in Figure 4, with

$$\begin{aligned} w_0(x, y) &= (tx, sy), & w_1(x, y) &= (sx, ty + (1 - t)), \\ w_2(x, y) &= (sx + (1 - s), ty), & w_3(x, y) &= (tx + (1 - t), sy + (1 - s)). \end{aligned}$$

We see that each w_i maps the unit square onto a corresponding *block* B_i in an affine fashion. The image of a line of slope c through the unit square is a line of slope $(t/s)c$ (for maps w_0, w_3) or slope $(s/t)c$ (for maps w_1, w_2) through block B_i.

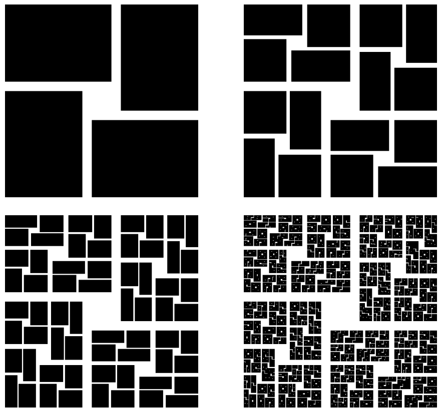

Figure 4: Construction for Cantor set projecting onto all directions.

We impose the conditions

$$\frac{1}{2} < s < 1 \quad \text{and} \quad (1/2)(1 - \sqrt{2s - 1}) < t < 1 - s.$$

These conditions first of all ensure that the attractor of the IFS given above will be totally disconnected, and it is easy to verify that the attractor is perfect. Thus the attractor will be a Cantor set. Furthermore, the condition that $1 - \sqrt{t} < s$ ensures that every line through $(1/2, 1/2)$ (the central point) will intersect one of the four blocks. The condition is derived by finding the requirement for line L_1 to have steeper slope than line L_2 (see Figure 5). This is essential for our construction. The idea (illustrated in Figure 5) is that if some line can pass through the unit square and miss all four blocks, then there must be some line through the central point which also misses all four blocks. This is true because of the symmetry in the figure.

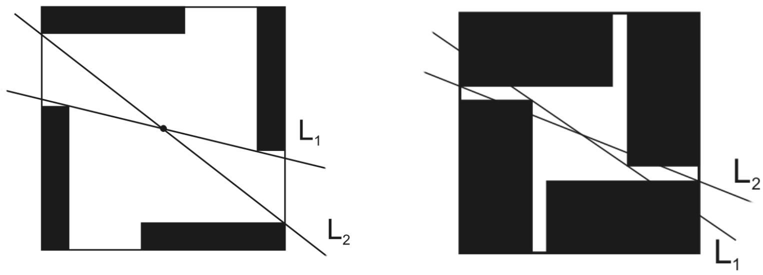

Figure 5: The conditions on s and t.

We will use induction to show that the attractor of this IFS (a Cantor set) projects onto a line segment in every direction. In fact, we show that any line through the unit square will intersect this Cantor set.

Let S_0 be the unit square $[0,1] \times [0,1]$, and let $S_1 = \widehat{w}(S_0)$ so that $S_1 = B_0 \cup B_1 \cup B_2 \cup B_3$.

For the basis of the induction, we must show that any such line L intersects S_1. However, this is guaranteed by the above conditions on s and t. Thus the basis for the induction is established.

Thus, suppose that any line through the unit square intersects $S_1, S_2, \ldots, S_n$ and consider $S_{n+1} = \widehat{w}(S_n)$. We see that S_{n+1} is naturally composed of four affine copies of S_n, one under each of the four maps w_j, and that each of these four copies is a subset of the corresponding block; that is, $w_j(S_n) \subset B_j$. Let L be some line through the unit square. Then L intersects at least one of the blocks B_j. We argue that L intersects $w_j(S_n) = S_{n+1} \cap B_j$. Now, $L \cap B_j$ is a line in B_j so $w_j^{-1}(L \cap B_j)$ is a line L' through the unit square (since w_j is affine and B_j is the range of w_j). However, we assumed that any line through the unit square intersects S_n, so $L' \cap S_n \neq \emptyset$. This implies that $L \cap w_j(S_n) \neq \emptyset$, or L intersects $w_j(S_n)$, so L intersects S_{n+1}.

Figure 6 illustrates both the line L and its intersection with S_{n+1} and the line L' and its intersection with S_n.

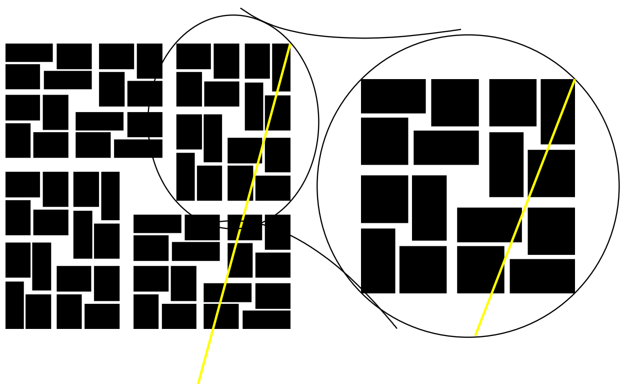

Figure 6: The intersection of L and S_{n+1} and L' and S_n.

Since each S_n intersects any line through the unit square, the projection of each S_n in any direction is the same as the projection of the unit square in the same direction, so is a line segment. Thus the Cantor set, which is the intersection of the S_n's, will also have the same projections as the unit square, so will project onto a line segment in every possible direction.

This construction can naturally be extended to $\mathbb{R}^n$, by using an IFS with $2n$ maps (one for each face of the unit cube in $\mathbb{R}^n$).

3 CONCLUDING COMMENTS

Falconer's (very good) book (reference [2]) has a chapter on the behavior of dimension under projections. This book and the references in it are excellent places to learn more about this subject. We briefly discuss some results which are related to the examples in this paper.

For $S \subset I\!R^2$, if $d = dim_H(S) < 1$ (where dim_H denotes the Hausdorff dimension), then the projection of S in almost every direction will have dimension equal to d and thus zero Lebesgue measure. Thus our Cantor set must have dimension at least one for any s and t which satisfy our conditions.

If $d = dim_H(S) \geq 1$, then the projection of S has positive Lebesgue measure for almost every direction. Notice that this does not mean that these projections will contain intervals (or be intervals).

As a final comment, we mention the "digital sundial" example. For each angle $\theta \in [0, \pi)$ let $F_\theta \subset I\!R$ be a Borel set. Then there exists a Borel set $F \subset I\!R^2$ so that for almost every θ, the projection of F in the direction of θ differs from F_θ by a set of measure zero! As Falconer comments, the 3D version of this result guarantees that there is some set F so that as the sun moves in the sky, the shadow cast by F gives the correct local time!

ACKNOWLEDGEMENTS. F. Mendivil gratefully acknowledges support from the National Sciences and Engineering Research Council of Canada (NSERC) in the form of a Discovery Grant. T. D. Taylor gratefully acknowledges research support from St. Francis Xavier University.

References

[1] M. F. Barnsley, *Fractals Everywhere*, Academic Press, New York, 1988.

[2] K. J. Falconer, *Fractal Geometry: Mathematical Foundations and Applicaitons*, John Wiley, New York, 1990.

[3] J. E. Hutchinson, Fractals and self-similarity, *Indiana Univ. Math. J.* **30** (1981) 713-747.

Department of Mathematics and Statistics, Acadia University, Wolfville, NS B4P 2R6, CANADA *franklin.mendivil@acadiau.ca*

Department of Mathematics, Statistics and Computer Science, St. Francis Xavier University, Antigonish, NS B2G 2W5, CANADA *ttaylor@stfx.ca*