Portfolio Optimization under Partial Uncertainty and Incomplete Information: A Probability Multimeasure-based Approach

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Abstract

Markowitz's work has had a major impact on academic research and the financial industry as a whole. The main idea of his model is risk aversion of average investors and their desire to maximise the expected return with the least risk. In this paper we extend the classical Markowitz's model by introducing a portfolio optmization model in which the underlying space of events is described in terms of a probability multimeasure. The notion of probability multimeasure allows to formalize the formalize of imprecise probability measure and incomplete information.

Keywords: Markowitz's model, set-valued measure, probability multimeasure, portfolio optimization, efficient frontier.

1 Portfolio Optimization

The theory of Portfolio Management and selection is a classical problem in Operation Research, Finance, Management, and Marketing. There are many applications of this theory to financial markets, technological change, strategic investments, among others. Portfolio Management gives the investor a quantitative methodoloy which will allow one to select the best available securities that will provide the expected rate of return and will also allow one to mitigate the risks.

The portfolio strategy approach dates back to the first mathematical formulation and model in the pioneering paper of the American economist Harry Markowitz [24]. Markowitz published his article in the Journal of Finance in

1952: his idea was revolutionary for several reasons. He formulated an innovative model that allows a joint quantitative evaluation of portfolio return and risk by considering returns and their correlations. He also provided a simple explanation of the concept of portfolio diversification: the riskiness of a portfolio depends on the correlations of its constituents, and not only on the average riskiness of its separate securities. And, finally, he formulated the financial decision-making process as an optimization problem. This allows the use of computationally efficient algorithms.

Markowitz's work is still in use sixty years later. His framework led to the concept of efficient portfolios: an efficient portfolio is expected to yield the highest return for a given level of risk or lowest risk for a given level of return. In his bi-criteria portfolio selection formulation he proposed a model where two conflicting and incommensurable criteria are to be optimized simultaneously, namely, the expected return and the risk and it is formulated as follows:

$$(P1) \qquad \max \sum_{j=1}^{m} \alpha_j \mathbb{E}(X_j) = \alpha \mathbb{E}(X)$$

$$\min \sum_{j=1}^{m} \sigma_{ij} \alpha_i \alpha_j = \alpha^T \Sigma(X) \alpha$$

Subject to:

$$\begin{cases} \sum_{i=1}^{m} \alpha_i = 1 \\ 0 \leq \alpha_i \leq 1 \end{cases}$$

where the first objective describes the expected return of the portfolio and the second objective its variance. The variables α_j takes into account the proportion to be invested in the stock (security) j, the term $\mathbb{E}(X_j)$ is the expected return of security j per unit, the symbol σ_{ij} denotes the covariance of the returns of securities j and k, and $\sigma(X)$ is the co-variance matrix.

This is a quadratic optimization problem with linear constraints: the set of all optimal solutions yield the so-called efficient frontier or efficient portfolios among which the decision maker will make a choice based on his/her preferences. This formulation is known in the literature as the mean-variance model: According to the so-called mean-variance optimization (MVO) problem, among all portfolios that achieve a particular return objective, the investor should choose the portfolio that has the smallest variance, the most efficient one with lower risk.

One way to simplify and solve the above problem consists of setting a level of risk avversion R and scalarize the problem as follow:

$$(P2) \qquad \max \sum_{j=1}^{m} \alpha_j \mathbb{E}(X_j)$$

subject to

$$\begin{cases} \sum_{i=1}^{m} \alpha_i = 1 \\ 0 \leq \alpha_i \leq 1 \\ \alpha^T \Sigma(X) \alpha \leq R \end{cases}$$

Eq. (P2) will be extended in the sequel of this paper by replacing the usual expectation with the expected value of a random variable w.r.t. a probability multimeasure.

In the literature of portfolio optmization several attempts have been made to extend the standard formulation of Markowitz's model by the inclusion of extra criteria. The determination of the optimal portfolio allocation is a complex and sophisticated process that cannot be completely described by the bi-criteria model proposed by Markowitz: the inclusion of additional criteria is becoming more and more common as well as the analysis of computationally efficient algorithms ([1, 5, 14, 27, 29]). For instance, in Zouponidis et al [34], the authors proposed to include in the portfolio selection the following fifteen criteria: i) gross book value per share, ii) capitalization ratio, iii) stock market value of each firm, iv) the marketability of each share, v) financial position progress, vi) dividend yield, vii) capital gain, viii) exchange flow ratio, ix) round lots traded per day, x) transaction value per day, xi) equity ratio, xii) price/earning ratio, xiii) structure ratio, xix) equity/debt ratio, and xv) return on equity. In Zopounidis and Doumpos [33] the authors propose to gather the above criteria into the following three main categories: a) the corporate validity objectives, b) the stocks acceptability, and c) the financial vigour criteria.

In this paper we move along a different perspective. Instead of extending the number of criteria to be considered in the optimization process, we introduce the notion of probability multimeasure and define a portfolio optimization problem with respect to this new object. The most important features of Markowitz model are the estimation of the expected value of each security and the correlation betweem them. This is typically done by assuming an underlying probability distribution of events that allows to estimate the above quantities. However very rarely does the decision maker have a complete knowledge of this probability distribution, as very often he is subject to incomplete or partial information. Other attempts have been made in the literature to mathematically describe this lack of complete information ([6, 7, 8, 11, 10, 30, 31]) and all of them rely on the imposition of lower and upper bounds for the underlying probability distribution. In this paper we move away from this stream and propose an innovative approach based on the notion of probability multimeasure: the probability of a certain event is not anymore a single number but a set of numbers instead. The name *probability multimeasure* is essentially due to the fact that the probability of an event takes multiple values. Several authors have studied the main properties of this extension of the classical notion of probability including, among others, Radon-Nikodým theorems, strong law of large numbers, etc. (see [2, 3, 12, 13]).

This paper proceeds as follows: Section 2 presents the main mathematical and statistical properties of this object. Section 3 is devoted to the introduction of a generalized notion of variance while Section 4 contains the formultion of the portfolio optimization problem with imprecise probability. This is a set-valued optimization problem. Section 5 shows how to scalarize this problem and obtain a parametrized family of portfolio optimization models.

2 Probability Multimeasures

2.1 Preliminaries on compact convex sets

For all of our set-valued objects we will use nonempty compact and convex sets, and thus we first give some preliminaries on the analysis involving such sets. We use $\mathcal{K}$ to denote the collection of all nonempty compact and convex subsets of $\mathbb{R}^d$. The standard operations of addition and scalar multiplication on $\mathcal{K}$ are given as

$$A + B := \{a + b : a \in A, b \in B\} \text{ and } \lambda A = \{\lambda a : a \in A\}.$$

A subset $A \in \mathcal{K}$ is *nonnegative* ($A \geq 0$) if $0 \in A$. Given $A \in \mathcal{K}$ and $p \in \mathbb{R}^d$, the *support function* is defined by

$$\operatorname{spt}(p, A) = \sup\{p \cdot a : a \in A\}$$

and A can be recovered from $\operatorname{spt}(\cdot, A)$ by

$$A = \bigcap_{\|p\|=1} \{x : x \cdot p \leq \operatorname{spt}(p, A)\}. \tag{2.1}$$

For all $\lambda \geq 0$ and $A, B \in \mathcal{K}$ we have that

$$\operatorname{spt}(p, \lambda A + B) = \lambda \operatorname{spt}(p, A) + \operatorname{spt}(p, B),\ \operatorname{spt}(p, -B) = \operatorname{spt}(-p, B). \tag{2.2}$$

We warn the reader that normally $\operatorname{spt}(p, -B) \neq -\operatorname{spt}(p, B)$. It is easy to show that

$$d_H(A, B) = \sup_{\|p\|=1} |\operatorname{spt}(p, A) - \operatorname{spt}(p, B)|, \tag{2.3}$$

where d_H is the usual *Hausdorff* metric on $\mathcal{K}$ (see [16]). The *norm* of $A \in \mathcal{K}$ is given by

$$\|A\| := \sup\{\|x\| : x \in A\} = \sup_{\|p\|=1} \operatorname{spt}(p, A) = d_H(A, \{0\})$$

and it is easy to show that this satisfies the usual properties of a norm.

We say that $A \subset \mathbb{R}^d$ is balanced if $\lambda A \subseteq A$ for all $|\lambda| \leq 1$. Furthermore, a *unit ball* in $\mathbb{R}^d$ is any balanced $\mathbb{B} \in \mathcal{K}$ with $0 \in \mathit{int}(\mathbb{B})$. Any unit ball defines a norm on $\mathbb{R}^d$ by using the Minkowski functional

$$\|x\| = \sup\{\lambda \geq 0 : \lambda x \in \mathbb{B}\}.$$

Whenever we have chosen such a set $\mathbb{B}$, we will always use this induced norm on $\mathbb{R}^d$. The *dual sphere* is given by

$$\mathbb{S}^* = \{y : \operatorname{spt}(y, \mathbb{B}) = 1\} \subset \mathbb{R}^d$$

and is also a nonempty compact set. Notice that since $\mathbb{B}$ is compact, for each $y \in \mathbb{S}^*$, there is some $x \in \mathbb{B}$ with $y \cdot x = 1$.

We can define a type of weak inner product on $\mathcal{K}$ by, given $A, B \in \mathcal{K}$,

$$\langle A, B \rangle = \int_{\mathbb{S}^*} \operatorname{spt}(p, A) \operatorname{spt}(p, B) \; dp, \tag{2.4}$$

where we integrate over $\mathbb{S}^*$ using normalized surface measure. It is easy to see that this "weak" inner product satisfies all the usual properties of an inner product except that for $\lambda < 0$ we have $\langle \lambda A, B \rangle = |\lambda| \langle -A, B \rangle$ which is generally not the same as $\lambda \langle A, B \rangle$. In particular, we can define another norm on $\mathcal{K}$ by

$$\|A\|_2 = \sqrt{\langle A, A \rangle} \tag{2.5}$$

and using this norm we get the Cauchy-Schwartz inequality

$$| \langle A, B \rangle | \leq \|A\|_2 \|B\|_2. \tag{2.6}$$

2.2 Multimeasures

Here we give only some very basic definitions and properties of multimeasures; for more information and proofs of the results see [2, 3, 15, 21]. A *set-valued measure* or *multimeasure* with values in $\mathcal{K}$ on the measurable space $(\Omega, \mathcal{A})$ is a set function $\phi : \mathcal{A} \to \mathcal{K}$ such that $\phi(\emptyset) = \{0\}$ and

$$\phi\left(\bigcup_i A_i\right) = \sum_i \phi(A_i) \tag{2.7}$$

for any sequence of disjoint sets $A_i \in \mathcal{A}$. The right side of (2.7) is the infinite Minkowski sum defined as

$$\sum_i K_i = \left\{ \sum_i k_i : k_i \in K_i, \sum_i |k_i| < \infty \right\}.$$

For our setting this sum also converges in the Hausdorff metric on $\mathcal{K}$. The *total variatoion* of a multimeasure ϕ is defined in the usual way as

$$|\phi|(A) = \sup \sum_i \|\phi(A_i)\|,$$

where the supremum is taken over all finite measurable partitions of $A \in \mathcal{A}$. The set-function $|\phi|$ defined in this fashion is a (nonnegative and scalar) measure on Ω. If $|\phi|(\Omega) < \infty$ then ϕ is of *bounded variation*.

A multimeasure ϕ is said to be *nonnegative* if $0 \in \phi(A)$ for all $A \in \mathcal{A}$. Nonnegativity implies monotonicity: if $A \subseteq B$ then $\phi(A) = \{0\} + \phi(A) \subseteq \phi(B \setminus A) + \phi(A) = \phi(B)$. This property indicates that nonnegative multimeasures are a natural generalization of (nonnegative) scalar measures. For a multimeasure ϕ and $p \in \mathbb{R}^d$, the *scalarization* ϕ^p defined by

$$\phi^p(A) = \operatorname{spt}(p, \phi(A)). \tag{2.8}$$

This gives a signed measure on Ω and a measure if ϕ is nonnegative.

A simple construction of a multimeasure is by integrating a *multifunction density* f with respect to a (scalar) measure μ:

$$\phi(A) = \int_A f(x)\ d\mu(x). \tag{2.9}$$

There are several approaches to defining this integral (see [4]). Since we only consider $f : \Omega \to \mathcal{K}$ we can define the integral in (2.9) as an element of $\mathcal{K}$ via support functions using the property (see [4, Proposition 8.6.2])

$$\mathrm{spt}\left(q, \int_\Omega f(x)\ d\mu(x)\right) = \int_\Omega \mathrm{spt}(q, f(x))\ d\mu(x),$$

which defines the set as in (2.1). If the multifunction f is nonnegative (that is, $0 \in f(x)$ for all x), then the resulting multimeasure will also be nonnegative. For more results on set-valued analysis see [4].

2.3 Probability Multimeasures

Definition 2.1 (probability multimeasure) Let $\mathbb{B} \subset \mathbb{R}^d$ be a unit ball. A $\mathbb{B}$-*probability multimeasure* (*pmm*) on $(\Omega, \mathcal{A})$ is a nonnegative multimeasure ϕ with $\phi(\Omega) = \mathbb{B}$.

Note that a probability multimeasure ϕ defines a parameterized family, ϕ^p for $p \in \mathbb{S}^*$, of probability measures. However, ϕ^p and ϕ^q are usually related and the relationship can be quite complicated.

One way to construct a pmm is by integrating a multifunction density $f : \Omega \to \mathcal{K}$ against a finite measure μ, as in (2.9). To guarantee a pmm, the simplest conditions on f are to assume that $f(x) \in \mathcal{K}$ is balanced for each x, $\|f(x)\| \leq C$ for some C and all x, and

$$0 \in int \int_\Omega f(x)\ d\mu = int(\mathbb{B}).$$

In general, it is difficult to choose a density to obtain a given $\mathbb{B}$; it is better to use the integral of the density to define $\mathbb{B}$.

Example 2.2 Let $0 < \alpha < \beta$ be given and define $F : [\alpha, \beta]^2 \subset \mathbb{R}^2 \to \mathcal{K}$ by $F((a, b)) = \mathcal{E}_{a,b}$, where the ellipse $\mathcal{E}_{x,y} = \{(x, y) \in \mathbb{R}^2 : a^2x^2 + b^2y^2 \leq 1\}$. Finally, let μ be any probability measure on $[\alpha, \beta]^2$ and define the Borel probability multimeasure ϕ

$$\phi(A) = \int_A F(a, b)\ d\mu(a, b).$$

The total mass of ϕ is clearly

$$\mathbb{B} = \int_{[\alpha,\beta]^2} F(a, b)\ d\mu(a, b) \in \mathcal{K}$$

and depends in a crucial way on α, β and μ.

In this context, a *random variable* on $(\Omega, \mathcal{A})$ is a Borel measurable function $X : \Omega \to \mathbb{R}$. The *expectation* of X with respect to a pmm ϕ is defined in the usual way as

$$\mathbb{E}_\phi(X) = \int_\Omega X(\omega)\ d\phi(\omega). \tag{2.10}$$

This integral can also be constructed using support functions (that is, using the ϕ^p) and each part of the decomposition $X = X^+ - X^-$ separately (since support functions work best with nonnegative scalars); see [15] for another approach. We say that X *has finite expectation* if $\mathbb{E}_\phi(X) \in \mathcal{K}$.

3 A Notion of Risk

In this section we introduce a notion of variance of a scalar random variable $X : \Omega \to \mathbb{R}$ w.r.t. a probability multimeasure ϕ. It is clear that one can not proceed with the well known defiinition of variance, as in this case X is a scalar number whilst $\mathbb{E}_\phi(X)$ is a subset of $\mathbb{R}^d$. To overcome this difficulty we introduce the following definition.

Definition 3.1 Given a scalar random variable $X : \Omega \to \mathbb{R}$ and a pmm ϕ, the variance of X is given by

$$\sigma^2_\phi(X) = \|\mathbb{E}_\phi(X^2)\|_2 - \|\mathbb{E}_\phi(X)\|_2^2.$$

The following results show that this definition complies with the most reasonable properties one could expect from a new definition of variance.

Theorem 3.2 *Suppose that $X(\omega) = k \in \mathbb{R}$ for all $\omega \in \Omega$. Then $\sigma^2_\phi(X) = 0$.*

Proof. By easy computations we get that

$$\begin{aligned}
\sigma^2_\phi(X) &= \|\mathbb{E}_\phi(K^2)\|_2 - < \mathbb{E}_\phi(K), \mathbb{E}_\phi(K) > \\
&= K^2 \|\phi(\Omega)\|_2 - K^2 < \mathbb{B}, \mathbb{B} > \\
&= K^2 - K^2 = 0.
\end{aligned}$$

■

Theorem 3.3 *Suppose $X : \Omega \to \mathbb{R}$ be a scalar random variable. Then $\sigma^2(X) \geq 0$.*

Proof. One can easily prove the following passages:

$$\begin{aligned}
\|\mathbb{E}_\phi(X^2)\|_2^2 &= \left\| \int_\Omega X^2 d\phi \right\|_2^2 \\
&= \int_{\mathbb{S}^*} \left[\text{spt} \left(\int_\Omega X^2 d\phi, p \right) \right]^2 dp
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{S}^*} \left[\int_\Omega X^2 d\phi^p\right]^2 dp \\
&= \int_{\mathbb{S}^*} \left[\mathbb{E}_{\phi^p}(X^2)\right]^2 dp \\
&\geq \int_{\mathbb{S}^*} \left[\mathbb{E}_{\phi^p}(X)\right]^4 dp \\
&\geq \left(\int_{\mathbb{S}^*} \left[\mathbb{E}_{\phi^p}(X)\right]^2 dp\right)^2 = \|\mathbb{E}_\phi(X)\|_2^4 \qquad (3.11)
\end{aligned}
$$

and this implies the thesis. ■

In a similar manner one could introduce the notion of covariance and then independence.

Definition 3.4 Given two scalar random variables X and Y, the covariance between X and Y is defined as follows (recall the inner product defined in (2.4))

$$\mathrm{cov}_\phi(X,Y) = \|\mathbb{E}_\phi(XY)\|_2 - < \mathbb{E}_\phi(X), \mathbb{E}_\phi(Y) > .$$

It is trivial to show that whenver $X = Y$ we have $\mathrm{cov}(X,Y) = \sigma^2(X)$. Furthermore, following the classical approach one can define a notion of covariance matrix for a random vector X.

Definition 3.5 Suppose that $X : \Omega \to \mathbb{R}^m$ be a random vector. Then the covariance matrix of X, denoted by $\Sigma(X)$, is defined as follows

$$[\Sigma_\phi(X)]_{ij} = \mathrm{cov}_\phi(X_i, X_j).$$

4 Portfolio Optimization using a Probability Multimeasure

Let $X = (X_1, X_2, \ldots, X_m)$ be a random vector, and suppose that X_i are positive random variables with finite expectation for all $i = 1...m$. We are now ready to state the formulation of the portfolio problem with respect to an underlying probability multimeasure structure as follows:

$$(P3) \qquad \max F(\alpha) := \sum_{i=1}^m \alpha_i \mathbb{E}_\phi(X_i)$$

subject to

$$\begin{cases} \sum_{i=1}^m \alpha_i = 1, \\ 0 \leq \alpha_i \leq 1, \\ \alpha^T \Sigma(X) \alpha \leq R, \end{cases}$$

where R describes the risk aversion of the decision maker. This is a set-valued optimization problem, as the objective function $F : \mathbb{R}^m \to \mathcal{K}$.

Theorem 4.1 *Let $\|\cdot\|_m$ and d_H be the m-dimensional Eucledean norm and the Hausdorff distance between compact sets, respectively. Then $F : (\mathbb{R}^m, \|\cdot\|_m) \to (\mathcal{K}, d_K)$ is Lipschitz and thus continuous.*

Proof. We see that

$$\begin{aligned} d_H\left(\sum_i \alpha_i^* \mathbb{E}_\phi(X_i), \sum_i \alpha_i \mathbb{E}_\phi(X_i)\right) &= \sup_{p\in\mathbb{S}^*} \left|\operatorname{spt}\left(p, \sum_i \alpha_i^* \mathbb{E}_\phi(X_i)\right) - \operatorname{spt}\left(p, \sum_i \alpha_i \mathbb{E}_\phi(X_i)\right)\right| \\ &= \sup_{p\in\mathbb{S}^*} \left|\sum_i (\alpha_i^* - \alpha_i) \operatorname{spt}(p, \mathbb{E}_\phi(X_i))\right| \\ &\le \sup_{p\in\mathbb{S}^*} \sum_i |\alpha_i^* - \alpha_i| \operatorname{spt}(p, \mathbb{E}_\phi(X_i)) \\ &\le \max_i \|\mathbb{E}_\phi(X_i)\| \sum_i |\alpha_i^* - \alpha_i|, \end{aligned}$$

from which the desired result easily follows. ■

Continuity of F and the compactness of the feasible set assure the existence of at least one optimal solution [17].

Theorem 4.2 *$F : \mathbb{R}^m \to \mathcal{K}$ is a non-negative set-valued function.*

Proof. To prove this, we should demonstrate that $0 \in F(\alpha)$ for all α. Since ϕ is a probability multimeasure and X_i is a positive scalar random variable, we know that $0 \in \mathbb{E}_\phi(X_i)$ for all $i = 1...m$. Then we have

$$0 = \sum_{i=1}^m \alpha_i 0 \in \sum_{i=1}^m \alpha_i \mathbb{E}_\phi(X_i) = F(\alpha).$$

■

The following result states a stability property of the expected value with respect to perturbation of the random variable X.

Theorem 4.3 *Suppose that X_1 and X_2 are two scalar random variables with finite expectations and defined on the same probability multimeasure space $(\Omega, \mathcal{A}, \phi)$. Then $d_H(\mathbb{E}_\phi(X_1), \mathbb{E}_\phi(X_2)|) \le \|X_1 - X_2\|_\infty$ where d_H denotes the Hausdorff distance between the compact sets $\mathbb{E}_\phi(X_1)$ and $\mathbb{E}_\phi(X_2)$.*

Proof. We compute that

$$\begin{aligned}
d_H(\mathbb{E}_\phi(X_1), \mathbb{E}_\phi(X_2)) &= \sup_{p\in\mathbb{S}^*} |\mathrm{spt}(p, \mathbb{E}_\phi(X_1)) - \mathrm{spt}(p, \mathbb{E}_\phi(X_2))| \\
&= \sup_{p\in\mathbb{S}^*} \left| \int_\Omega X_1 \; d\phi^p - \int_\Omega X_2 \; d\phi^p \right| \\
&= \sup_{p\in\mathbb{S}^*} \left| \int_\Omega (X_1 - X_2) \; d\phi^p \right| \\
&\le \sup_{p\in S^*} \int_\Omega |X_1 - X_2| \; d\phi^p \\
&\le \|X_1 - X_2\|_\infty,
\end{aligned}$$

as desired. ■

For more information about properties of set-valued function one can see [4]. The definition of maximum point is provided w.r.t. the following order: Given two sets A, B in $\mathcal{K}$, we say that A is greater than B, and we write $A \ge B$, if there exists a set $C \in \mathcal{K}$, with $0 \in C$, such that $A = B + C$. Note that $B = B + \{0\} \subset B + C = A$ in this case, so $B \le A$ implies $B \subseteq A$. The converse implication is not true in general. Our definition of the order on $\mathcal{K}$ is used in the following definition of maximum point.

Definition 4.4 Define the feasible set as follows

$$\mathcal{F} = \{\alpha \in \mathbb{R}^m : 0 \le \alpha_i \le 1, \alpha^T \Sigma(X)\alpha \le R\}.$$

We say that $\alpha^* \in \mathcal{F}$ is an optimal solution for the problem

$$\max_{\alpha\in\mathcal{F}} F(\alpha)$$

whenever $F(\alpha^*) \ge F(\alpha)$ for all $\alpha \in \mathcal{F}$ which means that for all $\alpha \in \mathcal{F}$ there exists a set $C(\alpha) \in \mathcal{K}$, with $0 \in C(\alpha)$, such that $F(\alpha^*) = F(\alpha) + C(\alpha)$.

5 Problem Scalarization

From now on let us suppose that $X : \Omega \to \mathbb{R}^m$ be a random vector. For each $p \in S^*$, let us consider the probability measure ϕ^p and construct the classical expectation and the variance of X_i w.r.t. ϕ^p, namely

$$\mathbb{E}_{\phi^p}(X_i) = \int_\Omega X_i(\omega) d\phi^p(\omega).$$

By introducing this scalar notion of expectation with respect to the scalarized probability distribution ϕ^p, the portfolio problem can be written as

$$(P4) \qquad \max F^p(\alpha) := \sum_{i=1}^m \alpha_i \mathbb{E}_{\phi^p}(X_i) = \mathrm{spt}(p, F(\alpha))$$

subject to

$$\begin{cases} \sum_{i=1}^m \alpha_i = 1, \\ 0 \le \alpha_i \le 1, \\ \alpha^T \Sigma(X) \alpha \le R. \end{cases}$$

The following theorem states a relationship between the optimal solutions of (P3) and (P4).

Theorem 5.1 *Let $\alpha^* \in \mathcal{F}$ be an optimal solution of (P3). The α^* solves the above problem (P4) for all $p \in S^*$.*

Proof. Since $\alpha^* \in \mathcal{F}$ is an optimal solution of (P3), then

$$F(\alpha^*) = F(\alpha) + C(\alpha)$$

for all $\alpha \in \mathcal{F}$, where $C(\alpha) \in \mathcal{K}$ and $0 \in C(\alpha)$. This means that

$$F^p(\alpha^*) = \mathrm{spt}(p, F(\alpha^*)) = \mathrm{spt}(p, F(\alpha) + C(\alpha)) \ge \mathrm{spt}(p, F(\alpha)) = F^p(\alpha).$$

■

Let us observe that in general the converse of the above theorem is not true. However, if $\alpha^* \in \mathcal{F}$ is an optimal solution of (P4) for all $p \in S^*$, then the inequalities in the proof show that $F(\alpha) \subseteq F(\alpha^*)$ for all $\alpha \in \mathcal{F}$. This does not necessarily imply that there is some $A \in \mathcal{K}$ with $F(\alpha) + A = F(\alpha^*)$.

5.1 An Example: The Case of Interval-Valued Probability Multimeasures

Let us discuss an example where ϕ takes interval values, that is $\phi(A) = [\phi_-(A), \phi_+(A)]$, $0 \in \phi(A)$ for all $A \in \mathcal{A}$, and $\phi(\Omega) = [-1.1]$. In this case if $p \in \mathbb{S}^*$ then either $p = 1$ or $p = -1$. When $p = 1$ then $\phi^p = \phi_+$ and problem (P4) can be stated as

$$(P4+) \qquad \max F^+(\alpha) := \sum_{i=1}^m \alpha_i \int_\Omega X_i(\omega) d\phi_+(\omega) = \sum_{i=1}^m \alpha_i \mathbb{E}_{\phi_+}(X_i)$$

subject to

$$\begin{cases} \sum_{i=1}^m \alpha_i = 1 \\ 0 \le \alpha_i \le 1 \\ \alpha^T \Sigma(X) \alpha \le R. \end{cases}$$

In a similar manner, when $p = -1$ then $\phi^p = -\phi_-$ and problem (P4) can be rewritten as

$$(P4-) \qquad \max F^-(\alpha) := \sum_{i=1}^m \alpha_i \int_\Omega X_i(\omega) d(-\phi_-)(\omega) = \sum_{i=1}^m \alpha_i \mathbb{E}_{-\phi_-}(X_i)$$

subject to

$$\begin{cases} \sum_{i=1}^m \alpha_i = 1 \\ 0 \le \alpha_i \le 1 \\ \alpha^T \Sigma(X) \alpha \le R. \end{cases}$$

In this particular case the solutions to problem (P3) can be found among the solutions to problems (P4+) and (P4-) that represent classical linear optimization problems subject to quadratic constraints.

As a numerical example, let us consider a space of events Ω composed by three different scenarios ω_1, ω_2, and ω_3 corresponding to economic boom, stagnation, and recession respectively. Associated with Ω there is a probability multimeasure ϕ which takes the following values $\phi(\omega_1) = [0, \frac{1}{2}]$, $\phi(\omega_2) = [-\frac{1}{2}, \frac{1}{2}]$, and $\phi(\omega_3) = [-\frac{1}{2}, 0]$. Trivially $0 \in \phi(\omega_j)$ for all $j = 1...3$ and $\phi(\Omega) = [0, \frac{1}{2}] + [-\frac{1}{2}, \frac{1}{2}] + [-\frac{1}{2}, 0] = [-1, 1]$. Furthermore in this case $\phi_-(\omega_1) = 0$, $\phi_-(\omega_2) = -\frac{1}{2}$, $\phi_-(\omega_3) = -\frac{1}{2}$ whilst $\phi_+(\omega_1) = \frac{1}{2}$, $\phi_+(\omega_2) = \frac{1}{2}$, $\phi_+(\omega_3) = 0$. Let X_i, $i = 1...m$, be a portfolio and let us suppose that $X_i(\omega_1) = X_i^*$, $X_i(\omega_2) = X_i^{**}$, and $X_i(\omega_2) = X_i^{***}$. In this case the above, model Eq. (P4+), reads as

$$\max F^+(\alpha) := \sum_{i=1}^{m} \alpha_i \int_\Omega X_i(\omega) d\phi_+(\omega) = \sum_{i=1}^{m} \frac{\alpha_i}{2}(X_i^* + X_i^{**})$$

subject to

$$\begin{cases} \sum_{i=1}^m \alpha_i = 1 \\ 0 \leq \alpha_i \leq 1 \\ \alpha^T \Sigma(X) \alpha \leq R \end{cases}$$

while model Eq. (P4-) becomes

$$(P4-) \qquad \max F^-(\alpha) := \sum_{i=1}^{m} \alpha_i \int_\Omega X_i(\omega) d(-\phi_-)(\omega) = \sum_{i=1}^{m} \frac{\alpha_i}{2}(X_i^{**} + X_i^{***})$$

subject to

$$\begin{cases} \sum_{i=1}^m \alpha_i = 1 \\ 0 \leq \alpha_i \leq 1 \\ \alpha^T \Sigma(X) \alpha \leq R. \end{cases}$$

These are simple linear programming problems and can be solved without any computational difficulty.

6 Conclusion

In this paper we have extended the classical portfolio optimization problem by using the notion of probability multimeasure. This is an elegant and sophisticated way to describe the notion of imprecise probability distribution or lack of complete information: associated with each event or scenario there is a set that describes a range of possible probability values. This is a first attempt to use the notion of probability multimeasure in porfolio management: future venues will include the extension of the notion of risk (Value at Risk, Entropic Risk, etc) as well as the analysis of dynamic models.

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