ON MINKOWSKI MEASURABILITY

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Abstract. Two "pathological" properties of Minkowski content are that countable sets can have positive content (unlike Hausdorff measures) and the property of a set being Minkowski measurable is quite rare. In this paper, we explore both of these issues.

In particular, for each $d \in (0, 2)$ we give an explicit construction of a countable Minkowski measurable subset of $\mathbb{R}^2$ of Minkowski dimension d and arbitrary positive Minkowski content. We also indicate how this construction can be extended to $\mathbb{R}^n$, to construct a countable subset with arbitrary positive Minkowski content of any dimension in $(0, n)$. Furthermore, we give an example of a strictly increasing C^1 function which takes a Minkowski measurable subset of $[0, 1]$ onto a set which is not Minkowski measurable but of the same dimension.

1. Introduction

For a subset $A \subset \mathbb{R}^N$ and $\epsilon > 0$, the *ϵ-dilation of A* is the set

$$A^\epsilon = \{x : \|x - a\| < \epsilon \text{ for some } a \in A\}.$$

For $0 \leq s \leq N$, the s-dimensional *upper* and *lower Minkowski contents* of A are defined by

$$\mathcal{M}^{*s}(A) = \limsup_{\epsilon \to 0} \frac{\mathcal{L}^N(A^\epsilon)}{\epsilon^{N-s}}$$

and

$$\mathcal{M}^s_*(A) = \liminf_{\epsilon \to 0} \frac{\mathcal{L}^N(A^\epsilon)}{\epsilon^{N-s}}.$$

In both of these $\mathcal{L}^N$ is the N-dimensional Lebesgue measure. We comment that often there is a normalization factor, but for our purposes we can leave this out, as we are mainly interested in if these are finite and positive and not in their exact value. The normalizing constants are necessary to have consistent values for the content if we consider a set $A \subset \mathbb{R}^n$ instead as $A \subset \mathbb{R}^m$, for some $m \geq n$.

The Minkowski contents behave in a similar manner to the Hausdorff measures in that if $0 \leq \alpha < \beta < \gamma \leq N$ and $0 < \mathcal{M}^{*\beta}(A) < \infty$ then $\mathcal{M}^{*\alpha}(A) = \infty$ and $\mathcal{M}^{*\gamma}(A) = 0$. Clearly the same behaviour is true for $\mathcal{M}^s_*$.

If $\mathcal{M}^s_*(A) = \mathcal{M}^{*s}(A)$, then their common value is the s-dimensional *Minkowski content* and is denoted by $\mathcal{M}^s(A)$. This happens precisely when the limit

$$\mathcal{M}^s(A) := \lim_{\epsilon \to 0} \frac{\mathcal{L}^N(A^\epsilon)}{\epsilon^{N-s}}$$

Key words and phrases. Minkowski dimension, Minkowski content, Minkowski measurability.

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exists.

The *upper Minkowski dimension* of A is defined by

$$d_{\bar{\mathcal{M}}}(A) = \inf\{s > 0 : \mathcal{M}^{*s}(A) = 0\} = \sup\{s \geq 0 : \mathcal{M}^{*s}(A) > 0\},$$

with a similar definition for the lower Minkowski dimension. The Minkowski dimensions are also known as the *box dimensions* in the literature, particularly in the fractals literature. The Minkowski contents are less well-known, as they have some undesirable properties. For more on Minkowski content, see [1, 2, 3].

Definition 1. *We say that a set $A \subset \mathbb{R}^N$ is* Minkowski measurable *if* $0 < \mathcal{M}_*^d(A) = \mathcal{M}^{*d}(A) < \infty$ *where d is the Minkowski dimension of A.*

Thus a Minkowski measurable set has positive and finite Minkowski content at the correct dimension, its Minkowski dimension.

One major drawback of the Minkowski contents are that they are not outer measures. In particular, they are not countably subadditive. This leads to the fact that the Minkowski dimensions are also not countably stable, that is, the dimension of a countable union can be strictly greater than the supremum of the dimensions of the individual parts. This "pathology" is one of those properties that we explore (and exploit) in our construction of Minkowski measurable countable sets with arbitrary dimension and content (in Sec. 2).

It is easy to see that $\mathcal{M}^{*s}(A) = \mathcal{M}^{*s}(cl(A))$ (similarly for lower content), and thus the Minkowski dimensions and contents don't distinguish between a set and its closure. For this reason, we deal only with closed sets. It is also easy to see that if A is unbounded then $\mathcal{L}^N(A^\epsilon) = \infty$ for all ϵ, and thus we deal only with compact sets.

For disjoint compact sets A_i, however, both the upper and lower content are finitely additive. This is easy to see as eventually the finitely many A_i^ϵ will be disjoint. Thus for this case, the lower and upper Minkowski dimensions are also finitely stable.

The Minkowski contents (and hence dimension) are well-behaved with respect to dilations. That is, if $f : \mathbb{R}^n \to \mathbb{R}^n$ scales all distances by a factor of r, then $\mathcal{M}^{*s}(f(A)) = r^s \mathcal{M}^{*s}(A)$ with a similar behaviour for $\mathcal{M}_*^s$. Clearly this means that such functions preserve the Minkowski dimensions. However, unlike Hausdorff measures, Minkowski content and measurability is not so well-behaved for more general types of smooth functions. In Sec. 3 we give an example of this in one dimension.

Subsets of $\mathbb{R}$

For compact subsets of $\mathbb{R}$ there is an elegant characterization of both the Minkowski dimensions and which sets are Minkowski measurable. This begins with a nice geometric observation. Take a compact $A \subset R$. If $\mathcal{L}^1(A) > 0$, then it is easy to see that A has dimension one and is Minkowski measurable. Thus, we assume that $\mathcal{L}^1(A) = 0$. We let (α_i, β_i) be the bounded open sets in $\mathbb{R} \setminus A$, labeled in order of decreasing size, and let $a_i = \beta_i - \alpha_i$. For simplicity, assume that $A \subset [0, \sum_i a_i]$. For a given $\epsilon > 0$, let N_ϵ satisfy $a_{N_\epsilon+1} \leq 2\epsilon \leq a_{N_\epsilon}$. Then we have (see also [4, 3])

$$V(\epsilon) := \mathcal{L}^1(A^\epsilon) = 2(N_\epsilon + 1)\epsilon + \sum_{i \geq N_\epsilon+1} a_i. \tag{1}$$

To see this, we just notice that if the gap (α_i, β_i) is small enough (smaller than 2ϵ), then it is entirely contained in A^ϵ. Otherwise, A^ϵ only contains an interval of length 2ϵ around a given endpoint of the gap.

This formula (1) relates the asymptotics of $V(\epsilon)$ (and hence the Minkowski contents) to the asymptotics of the sequence of gap lengths, $\{a_i\}$. In particular, *only* the gap lengths matter, not the geometric placement of the gaps on the real line. Thus, we speak of a compact subset of $\mathbb{R}$ in terms of its sequence of gap lengths, which we always list in non-increasing order.

For $a_i = i^{-p}$, with $p > 1$ fixed, it is a relatively straightforward calculation to show that this leads to a Minkowski dimension (both upper and lower) of $1/p$. In fact, with a bit more work, it can be shown that any set with gap lengths i^{-p}, $p > 1$ will be Minkowski measurable at dimension $1/p$. A remarkable theorem of Lapidus and Pomerance [5] (and given a new proof in [6]) shows that all Minkowski measurable subsets of $\mathbb{R}$ are of this form. More precisely

Theorem 1. [5, 6] *Let $A \subset \mathbb{R}$ be compact with Lebesgue measure zero and let $\{a_i\}$ be the sequence of gap lengths for A. Then A is Minkowski measurable of dimension $d \in (0, 1)$ iff $a_i \sim C\, i^{-1/d}$ for some $C > 0$.*

Their results give much more, including a relation between the Minkowski content and the constant C (in [5], Lapidus and Pomerance were primarily interested in questions involving the spectrum of the Laplacian and this particular result was a necessary step in that study).

Fix $p > 1$ and set $a = i^{-p}$. Since the gap lengths are all that matter, we can construct a countable subset $E_p \subset [0, \sum_i a_i]$ with gap lengths $\{a_i\}$ by setting

$$E_p = \{\sum_{i \le n} a_i : n = 0, 1, 2, \dots, \} \cup \{\sum_i a_i\}.$$

This set E_p is Minkowski measurable with dimension $1/p$ by Theorem 1 and hence $0 < \mathcal{M}^{1/p}(E_p) < \infty$. Using E_p we can construct a Minkowski measurable subset of $\mathbb{R}$ of any desired content; we simply scale E_p by the appropriate amount r so that $r^{1/p}\mathcal{M}^{1/p}(E_p)$ is the desired content.

Our aim in the next section is to do the same in $\mathbb{R}^N$. That is, our aim is to construct a countable subset $E \subset \mathbb{R}^N$ which has Minkowski dimension $d \in (0, N)$ and is Minkowski measurable at this dimension. Then by a simple scaling we can obtain any positive and finite content.

2. Countable subsets of arbitrary content

The situation in $\mathbb{R}^2$ is more complicated due to the geometry in $\mathbb{R}^2$; there is no formula corresponding to (1) for $V(\epsilon)$ in terms of "gap lengths" and hence no simple characterization of Minkowski measurability. For certain classes of "cut-out" sets, a similar type of analysis can be done, but clearly "cut-out" sets in higher dimensions form a small portion of the collection of all possible compact sets.

We content ourselves here with providing for each $d \in (0, 2)$ a construction of a countable set $E_d \subset \mathbb{R}^2$ with $0 < \mathcal{M}^d(E_d) < \infty$. For $d \in (0, 1)$, we could simply mimic the construction in $\mathbb{R}$ and place the points along a line in $\mathbb{R}^2$ with "gaps" $i^{-1/d}$. However, for $d \in [1, 2)$, this construction will clearly not work as the sum of the gap lengths would be infinite. What is necessary is that the countably many points accumulate to a limit in a more "two-dimensional" way, so that the "sum of the gaps" can be finite even when $d \in [1, 2)$. This is the philosophy behind our construction.

We give explicit details for the construction and for the proof of the Minkowski measurability in $\mathbb{R}^2$. The situation in $\mathbb{R}^N$ is similar and we briefly discuss the situation in $\mathbb{R}^N$, but the pattern is analogous to that in $\mathbb{R}^2$ so we leave out the details.

2.1. **2-dimensional case.** Throughout this construction, we have a fixed $p > 0$. Our countable set E_d is constructed as a point at the origin together with a subset of points on a countable collection of concentric circles, S_n, centered at $(0, 0)$ and with the radius of S_n being n^{-p}. For each of the circles S_n, we choose a finite set of equally spaced points on S_n to belong to the countable set E_d, see Fig. 1. Notice that in our illustration the points towards the middle merge together, as we have to render them with positive size. This is important to note since in E_d^ϵ each point will have an ϵ-ball surrounding it and these balls always merge in the center of E_d.

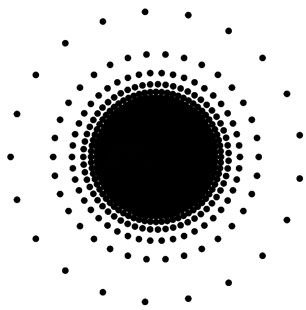

FIGURE 1. Illustration of countable set.

More formally, noting that the distance between S_n and S_{n+1} is $n^{-p} - (n+1)^{-p}$, we set

$$K_n = \left\lfloor \frac{2\pi}{\arccos\left(1 - \frac{(n^{-p}-(n+1)^{-p})^2}{2(n^{-p})^2}\right)} \right\rfloor = \left\lfloor \frac{2\pi}{\arccos\left(1 - \frac{1}{2}(1-(\frac{n}{n+1})^p)^2\right)} \right\rfloor \tag{2}$$

to be the number of equally spaced points we choose on S_n. For simplicity, we always choose one of the points to be on the positive x-axis, which then completely specifies the locations of all the other points. This choice of K_n results in the spacing between the points on S_n being asymptotically the same as the distance between S_n and S_{n+1}, and is obtained by solving for the angle which results in a spacing of $n^{-p} - (n+1)^{-p}$.

Having chosen K_n equally spaced points on the circle S_n, we define E_d to be the set containing the origin and all these K_n points for each S_n. Clearly E_d is countable and compact.

We now determine the Minkowski dimension of E_d and show that E_d has positive and finite content in this dimension. These results come from a careful estimation of the area of E_d^ϵ, and so we proceed to discuss this with a fixed value of ϵ.

We first make some elementary but useful observations. In general, for a given ϵ there will be three different types of behaviours of the ϵ-balls. The "outermost" ones will all be disjoint and this will happen for those points on S_n for small n. The "innermost" ones will all overlap and form a dense core around the origin, in fact, approximately a disk. Finally, those in the "middle" will partially overlap both with other ϵ-balls on the same S_n and with those on S_{n+1} and S_{n-1}. In this "middle" range, there are uncovered gaps left in the annulus between S_n and S_{n+1}, see Fig. 2. We now determine these three ranges of n for a given ϵ.

The first range is the simplest to determine. We simply need 2ϵ to be smaller than the spacing between the points. This occurs when $2\epsilon \leq n^{-p} - (n+1)^{-p}$. Estimating with the Mean Value theorem, we obtain that the cutoff is when

$$n \leq \left(\frac{p}{2\epsilon}\right)^{\frac{1}{p+1}} \leq n+1 \quad \Rightarrow \quad N_\epsilon = \left\lfloor \left(\frac{p}{2\epsilon}\right)^{\frac{1}{p+1}} \right\rfloor. \tag{3}$$

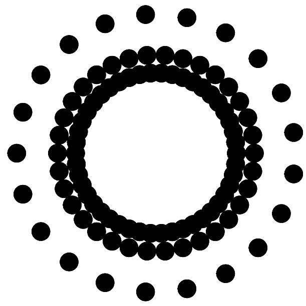

FIGURE 2. Balls just starting to overlap.

Thus all ϵ-balls for points on S_i with $1 \leq i \leq N_\epsilon$ will be disjoint. The area of this portion of E_d^ϵ is

$$\pi\epsilon^2 \sum_{i=1}^{N_\epsilon} K_i. \tag{4}$$

Before we start obtaining an asymptotic estimate of this area, we briefly mention two facts which are very useful to keep in mind. First, if we have an increasing function $\phi(x)$ then we can use the estimate

$$\int_1^{N_\epsilon} \phi(x)\ dx \leq \sum_{i=1}^{N_\epsilon} \phi(i) \leq \int_2^{N_\epsilon} \phi(x)\ dx + \phi(N_\epsilon + 1).$$

Thus the sum and the integral are asymptotic as long as the term $\phi(N_\epsilon + 1)$ is of lower order than either of the integrals. This could cause a problem if N_ϵ grows too quickly, but we will see that we won't have this problem.

The second useful observation is that if $0 < \alpha < \beta$ then

$$\int_1^N c_1 x^\alpha + c_2 x^\beta\ dx \sim \int_1^N c_1 x^\beta\ dx.$$

This is a rather obvious fact, perhaps, but in our situation we use it repeatedly. Many of our estimates are based on Taylor expansions and thus the lower order terms are of a strictly lower polynomial order and so manipulations with sums and integrals are easily justified.

The first step in estimating (4) is to notice that

$$K_i \approx \frac{2\pi}{p}(i+1) + \frac{\pi(p-1)}{p} + \cdots \tag{5}$$

by a Taylor expansion of the function

$$\frac{2\pi}{\arccos\left(1 - \frac{1}{2}(1-(1-x))^p)^2\right)}$$

at $x = 0$. Notice that we only need this expansion for $x > 0$. This gives that

$$K_{N_\epsilon} \sim \frac{2\pi}{p}\left(\frac{p}{2}\right)^{\frac{1}{p+1}} \epsilon^{\frac{-1}{p+1}} + \frac{\pi(p+3)}{p} + \cdots.$$

We will see below that the integral has order $\epsilon^{\frac{-2}{1+p}}$, so that our integral is indeed asymptotic to the sum.

Recalling from (3) that $N_\epsilon \sim (p/2\epsilon)^{1/(p+1)}$ we see

$$\pi\epsilon^2 \sum_{i=1}^{N_\epsilon} K_i \sim \pi\epsilon^2 \int_1^{N_\epsilon} \frac{2\pi}{p} x \, dx \sim \frac{\pi^2}{p} \left(\frac{p}{2}\right)^{\frac{2}{p+1}} \epsilon^{2-\frac{2}{p+1}}.$$

Thus, the area of the "outermost" range is asymptotically of order $\epsilon^{2-2/(p+1)}$, which would give a Minkowski dimension of $2/(p+1)$. We will see that the other two parts have the same asymptotic order so this is indeed the dimension of E_d.

Thus we have proved:

Lemma 1. *The contribution of the "outermost" range to the area of E_d^ϵ is asymptotic to*

$$\frac{\pi^2}{p} \left(\frac{p}{2}\right)^{\frac{2}{p+1}} \epsilon^{2-\frac{2}{p+1}}.$$

For the "innermost" range, we need to estimate n so that $i \geq n$ implies that the entire area between S_i and S_{i+1} is covered by the ϵ-balls. We use a conservative estimate, which will make this n perhaps larger than it really is. Our estimate is based on approximating the annulus between S_n and S_{n+1} as a thin rectangle (see Fig. 3). We also assume that the centers of the ϵ-disks are separated by the same distance as the thickness, D, of this thin rectangle (also an approximation). Given this, the area in the rectangle is completely covered when $\epsilon = \sqrt{2}D/2 = D/\sqrt{2}$. Since the separation on S_n is $n^{-p} - (n+1)^{-p}$, this then yields an estimate of the cutoff at

$$M_\epsilon = \left\lceil \left(\frac{p}{\sqrt{2}\epsilon}\right)^{\frac{1}{p+1}} \right\rceil. \tag{6}$$

That is, for $n \geq M_\epsilon$ all the ϵ-balls on S_n will merge together into one larger ball. The area of this ball, the "inner" area A_{inner}, will thus be

$$\pi(M_\epsilon^{-p} + \epsilon)^2 \approx \pi \left[\left(\frac{\sqrt{2}}{p}\right)^{\frac{2p}{p+1}} \epsilon^{2-\frac{2}{p+1}} + 2\left(\frac{\sqrt{2}}{p}\right)^{1-\frac{1}{p+1}} \epsilon^{2-\frac{1}{p+1}} + \epsilon^2 \right]. \tag{7}$$

Thus again we see that the leading term is of order $\epsilon^{2-\frac{2}{p+1}}$, so we have:

Lemma 2. *The contribution of the "innermost" range to the area of E_d^ϵ is asymptotic to*

$$\pi \left(\frac{\sqrt{2}}{p}\right)^{\frac{2p}{p+1}} \epsilon^{2-\frac{2}{p+1}}.$$

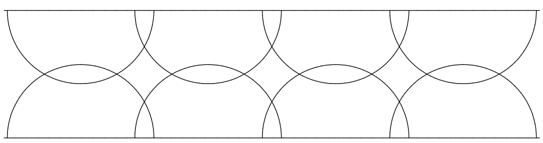

FIGURE 3. Estimating the cutoff for the "innermost" range.

Notice that $M_\epsilon \approx 2^{\frac{1}{2+2p}} N_\epsilon$, so that the size of the index set for the "middle" range, that is i in the range $N_\epsilon \leq i \leq M_\epsilon$, increases without bound. This is unfortunate, as if the size of the "middle" range was always bounded we could ignore it, since it would only account for a vanishingly small proportion of the total area. It is easy to find an upper bound on the area corresponding to this "middle" range as

$$\pi N_\epsilon^{-2p} - \pi M_\epsilon^{-2p} \sim 2\pi p^{-\frac{2p}{p+1}} \left(2^{\frac{2p}{p+1}} - 2^{\frac{p}{p+1}}\right) \epsilon^{2-\frac{2}{p+1}}.$$

This is of the same order as the other two parts. In fact, the area corresponding to this "middle" range turns out to be of this order in ϵ, and thus we must carefully estimate it. This is the most delicate part of the estimation because of the partial overlaps.

The underlying idea of our approach is to estimate, for each annulus, the proportion of the area of the annulus which is covered as a function of i in the "middle" range $N_\epsilon \le i \le M_\epsilon$. The annulus is divided into "cells" and so we do this as both a function of the index of the cell and of the annulus. For a fixed ϵ, this results in a double sum over the cells and the annuli in the range; we argue that as $\epsilon \to 0$ (so $N_\epsilon \to \infty$), this double sum converges to an integral in such a way that the Minkowski content exists.

To simplify our explanation, we will give our illustrations and our calculations as if the annulus between S_i and S_{i+1} were a thin rectangle. Clearly this is not the case; however, we do this for ease of understanding and clarity. After we deal with this simplified and approximate situation, we discuss how things change in the actual situation. We will not derive formulas for the "actual" situation, but our discussion should convince the reader that such formulas do indeed exist. As existence is all we are proving, this is sufficient.

By (5), we see that $K_n \sim (2\pi/p)(n+1)$ and so if $p < 2\pi$ then $K_{n+1} \ge K_n + 1$. What this means is that the points on S_{n+1} are "shifted" with respect to those on S_n, as in Fig. 4, and thus the ϵ-balls overlap in many different ways. As n increases, this results in these "overlaps" exhibiting the range of all the possible types of "shifts". If $p \ge 2\pi$, then there is a "shift" only between some fraction of the transitions from S_n to S_{n+1}, so most of the overlaps are more like those in Fig. 3 than those in Fig. 4. We discuss these two cases separately. Notice that since all the disks are of radius ϵ, the area of intersection of any two disks will depend only on the distance between their centers. In fact, the area of this "lens" is

$$\epsilon^2 \left[2 \arccos\left(\frac{D}{2\epsilon}\right) - 2\frac{\sqrt{\epsilon^2 - D^2/4}}{\epsilon} \frac{D}{2\epsilon} \right],$$

where ϵ is the radius and D is the distance between the centers.

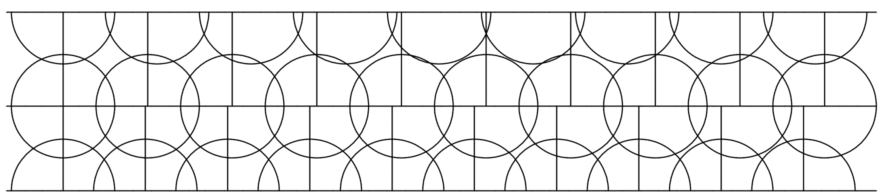

FIGURE 4. Two rows of the "middle" range with the shifts and "cells".

The case where $p < 2\pi$

First we deal with the case $p < 2\pi$ for which $K_{n+1} \geq K_n + 1$. We refer to Fig. 5 for illustrations. The height of each "cell" between S_n and S_{n+1} is $d = n^{-p} - (n+1)^{-p}$ while the width is $d' = (n+1)^{-p} - (n+2)^{-p}$. We see that

$$\xi := \frac{d'}{d} = \frac{(n+1)^{-p} - (n+2)^{-p}}{n^{-p} - (n+1)^{-p}} = \frac{1 - \left(1 - \frac{1}{n+2}\right)^p}{\left(1 + \frac{1}{n}\right)^p - 1} \sim \frac{n}{n+2} = 1 - \frac{2}{n+2}.$$

For a given spacing d, we see that for $n_d \approx (p/2d)^{1/(p+1)}$ we have $n_d^{-p} - (n_d+1)^{-p} \approx d$ and thus

$$\xi \sim 1 - 2\left(\frac{2}{p}\right)^{\frac{1}{p+1}} d^{\frac{1}{p+1}}$$

as the asymptotic ratio $d' : d$. In a similar way we see that with $d_n = n^{-p} - (n+1)^{-p}$, then

$$d_n \sim \frac{p}{n^{p+1}}. \tag{8}$$

Now we examine some of the possibilities. We give explicit formula for two of the cases just to indicate what form the formulas take. The exact formulas are not important, only the fact that they are well-behaved as a function of their parameters in the range of interest. The variable t represents the "shift" in the position of the point on S_n; t is measured from the upper left corner and is in the range $0 \leq t \leq d'/2$.

For d "large", that is, for an annulus towards the start of the "middle" range and for a "shift" t close $d'/2$ (so close to the center of the cell), we have the situation as illustrated in the first image in Fig. 5. In this case, the area which is covered is

$$\begin{aligned} &\pi\epsilon^2 - \epsilon^2 \int_{t/\epsilon}^{1} \sqrt{1-x^2}\,dx - \epsilon^2 \int_{(d'-t)/\epsilon}^{1} \sqrt{1-x^2}\,dx \\ &- \epsilon^2 \left[2\arccos\left(\frac{\xi d}{2\epsilon}\right) - \frac{\sqrt{\epsilon^2 - d^2\xi^2/4}}{\epsilon}\frac{d\xi}{\epsilon} \right]. \end{aligned} \tag{9}$$

The two integral terms correct for the fact that the ends of the upper disk are cutoff since $2\epsilon > d > d'$, as can be seen in the first image in Fig. 5.

On the other hand, for any d, if t is close to 0 then we have the situation as illustrated in the third image in Fig. 5. Here, the area which is covered is given by

$$\begin{aligned} &\pi\epsilon^2 - \epsilon^2 \int_{t/\epsilon}^{(t+(d-d'))/\epsilon} \sqrt{1-x^2}\,dx \\ &- \epsilon^2 \left[2\arccos\left(\frac{d}{2\epsilon}\right) - \frac{\sqrt{\epsilon^2 - d^2/4}}{\epsilon}\frac{d}{\epsilon} \right] \\ &- \epsilon^2 \left[2\arccos\left(\frac{\xi d}{2\epsilon}\right) - \frac{\sqrt{\epsilon^2 - \xi^2 d^2/4}}{\epsilon}\frac{\xi d}{\epsilon} \right] \\ &- \epsilon^2 \left[2\arccos\left(\frac{\sqrt{t^2+d^2}}{2\epsilon}\right) - \frac{\sqrt{\epsilon^2 - \sqrt{t^2+d^2}/4}}{\epsilon}\frac{\sqrt{t^2+d^2}}{\epsilon} \right]. \end{aligned} \tag{10}$$

The integral term is to compensate for the fact that the spacing between the points on S_n is d while the width of the cell is $d' = \xi d < d$. Thus, rather than the top disks contributing $\pi\epsilon^2/2$ they contributes a bit less (minus the overlaps, of course).

We are primarily interested in the ratio of the area which is covered, so we divide the above by the area of a cell, $d^2\xi$.

For a fixed ϵ there are only a finite number of different t values which are represented in any annulus and there are also only a finite number of different values of the parameter d. In fact, $M_\epsilon - N_\epsilon \approx (2^{\frac{1}{2p+2}} - 1)N_\epsilon$ different d values and at

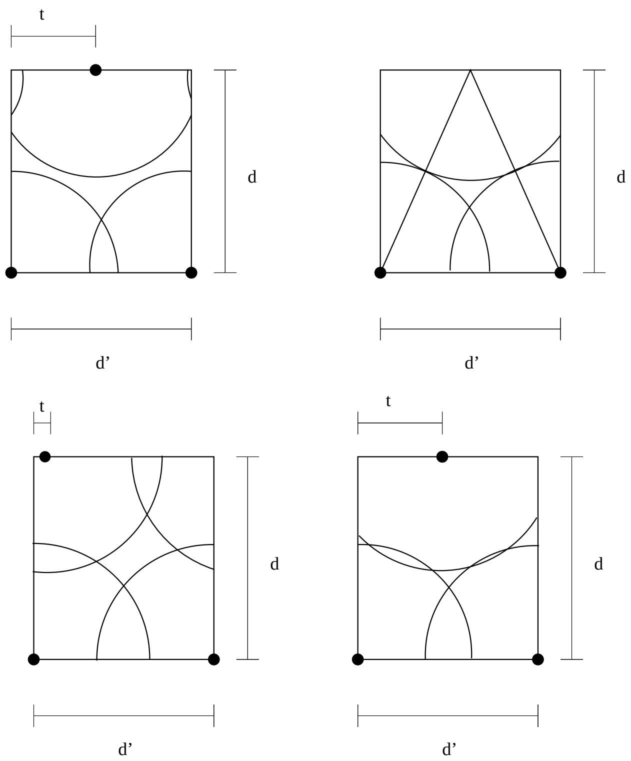

FIGURE 5. Illustration of types of overlaps for "middle" range.

most K_i different t values on a given annulus. As $\epsilon \to 0$, both d and t assume more values until, in the limit, they fill out their range. The range of t is simply $0 \leq t \leq d'/2 = \xi d/2$. The range of d is obtained by noting that the "middle" range corresponds to those annuli whose width is between 2ϵ and $\sqrt{2}\epsilon$ and thus $\sqrt{2}\epsilon \leq d \leq 2\epsilon$.

Notice that everything scales with ϵ. If we reduce ϵ by some factor ρ, then the "middle" range is shifted and both the parameters d and t simply scale exactly by a factor of ρ. That is, there are cells in this new range which correspond to scaled versions of the cells in the old range. This point is crucial in showing convergence as this is what leads to being able to approximate the double sum with an integral. It also means that we can scale out the dependence on ϵ and obtain the limiting ranges

$$\sqrt{2} \leq d \leq 2, \quad \text{and} \quad 0 \leq t \leq d'/2. \tag{11}$$

We now describe the function $\phi(d,t)$. This function gives the proportion of the area in a cell which is covered, with "width" parameter d and "shift" t. We could give an explicit expression for ϕ, but the specific details are not very important (especially since, at the moment, we are only describing the approximation where we think of the annuli as rectangles). What is important is to understand that such an expression does exist and that the function is a differentiable function of its arguments. The differentiability can be seen either from geometrical considerations or by noting that the formula is piecewise defined by expressions of the form (9) and (10) and these expressions are differentiable.

For $\sqrt{2} \leq d \leq 2/\sqrt{1+\xi^2/4}$ there are only intersections of the type illustrated in Fig. 5 images a) and c). If $0 \leq t \leq \sqrt{4-d^2}$, then the intersection is of the type illustrated in Fig. 5 image c). If $\sqrt{4-d^2} < t \leq d'/2$, then we have the situation illustrated in image a) of that figure. Thus, for these ranges of d and t, we have use these figures to derive expressions for ϕ.

If $2/\sqrt{1+\xi^2/4} < d \leq 2$, then we have intersections of the type illustrated in Fig. 5 images c) and d). For $0 \leq t \leq d' - \sqrt{4-d^2}$ we have the situation from Fig. 5 c) and for $d' - \sqrt{4-d^2} < t \leq d'/2$ we have the situation illustrated in d).

The resulting piecewise expression for $\phi(d,t)$ is rather long and messy, but composed of differentiable functions which stitch together differentiably at the boundaries. Furthermore, since we can scale the ϵ dependence out, $\phi(d/\epsilon, t)$ represents a dimensionless proportion of the area of a cell which is covered. The proportion is always strictly positive; that is, there is some $\eta > 0$ so that

$$0 < \eta \leq \phi(d/\epsilon, t) \leq 1 \tag{12}$$

for all t, d.

In the end, for a fixed ϵ we obtain an expression of the form

$$\sum_{N_\epsilon \leq i \leq M_\epsilon} \xi_i d_i^2 \sum_{j \text{ over the "cells"}} \phi(d_i/\epsilon, t_j). \tag{13}$$

The inner sum is easily converted to one over the various shifts, t_i. In fact, for the annulus between S_n and S_{n+1}, we have rational shifts of the form $\ell/(K_n K_{n+1})$ for various values of ℓ and these uniformly and densely fill the interval $[0, d'/2]$ as $\epsilon \to 0$. This is a result of the asymptotic estimate $K_n \sim (2\pi/p)(n+1)$ and so $K_{n+1} - K_n \approx (2\pi/p)$.

We will transform the outer sum into an integral over the values of d in the range $\sqrt{2} \leq d \leq 2$ by a suitable change of variable. Notice that we have to be careful since the values of d_i are not uniformly distributed over this range, as there are more "small" values of d_i than "large" values. However, the change of variable will naturally account for this.

Returning to our sum (13), we first notice that the inner sum can be written as

$$\sum_{j \text{ over the "cells"}} \phi(d_i/\epsilon, t_j) = K_i \widehat{\phi}(d_i/\epsilon),$$

where $\widehat{\phi}$ is the average value of ϕ over this annulus. This average value only depends on the quantity d/ϵ. Continuing, we use the asymptotic expressions for d_i, ξ, and K_i to obtain (after some simplification)

$$\sum_{i=N_\epsilon}^{M_\epsilon} \left[1 - \frac{2^{\frac{p+2}{p+1}}}{i}\right] \frac{2\pi p}{i^{2p+1}} \widehat{\phi}_i. \tag{14}$$

Now, using an integral approximation, we get that this is asymptotic to

$$\int_{N_\epsilon}^{M_\epsilon} \left[1 - \frac{2^{\frac{p+2}{p+1}}}{x}\right] \frac{2\pi p}{x^{2p+1}} \widehat{\phi}_x \, dx. \tag{15}$$

We now use the change of variables $y = (p/\epsilon)x^{-p-1} = d/\epsilon$. The reason for this change of variable is that then $x = N_\epsilon$ becomes $y = 2$ and $x = M_\epsilon$ becomes $y = \sqrt{2}$. After this change of variable, our integral is

$$\frac{2\pi p}{p+1} \left(\frac{\epsilon}{p}\right)^{2-\frac{2}{p+1}} \int_{y=\sqrt{2}}^{y=2} \left[1 - 2^{\frac{p+2}{p+1}} \left(\frac{\epsilon}{p}\right)^{\frac{1}{p+1}} y^{\frac{1}{p+1}}\right] y^{1-\frac{2}{p+1}} \widehat{\Phi}(y) \, dy. \tag{16}$$

As we have no specific form for $\widehat{\phi}(x)$, we don't have a specific form for $\widehat{\Phi}(y)$. However, $\widehat{\phi}$ is only a function of $d/\epsilon = y$, so $\widehat{\Phi}$ is only a function of y. Further, our bound (12) gives $0 < \eta \leq \widehat{\Phi}(y) \leq 1$, so that the value of the integral is bounded away from zero. Thus, as ϵ tends to zero, the integral converges to the constant value

$$C_p := \int_{y=\sqrt{2}}^{y=2} y^{1-\frac{2}{p+1}}\, \widehat{\Phi}(y)\; dy \tag{17}$$

and so the area of this "middle" range is asymptotically equal to

$$C_p\, \frac{2\pi p^{\frac{2}{p+1}-1}}{p+1}\, \epsilon^{2-\frac{2}{p+1}}, \tag{18}$$

in particular it is of asymptotic order $\epsilon^{2-\frac{2}{p+1}}$, as desired.

Lemma 3. *In the case that $p < 2\pi$, the contribution of the "middle" range to the area of E_d^ϵ is asymptotically equal to*

$$C_p\, \frac{2\pi p^{\frac{2}{p+1}-1}}{p+1}\, \epsilon^{2-\frac{2}{p+1}},$$

where C_p is as given in (17).

The case where $p \geq 2\pi$

If $p \geq 2\pi$, then much of the time $K_n = K_{n+1}$ and so we don't get the same types of "shifts" between the points on successive S_n. In this case, we can do a similar analysis as above, except for most of the annuli the situation is as illustrated in Fig. 3 and only for a small proportion are there "shifts". We briefly discuss the differences that this introduces in the analysis.

Because of the two different behaviours, we will have two different functions: $\phi(d/\epsilon, t)$ for the "shifts" and $\psi(d/\epsilon, t)$ for the situation where there are no "shifts". As before (in equation (13)), we sum over the annuli and cells and obtain a sum

$$\begin{aligned} &\sum_{N_\epsilon \leq i \leq M_\epsilon} \xi_i d_i^2 \sum_{j \text{ over " shift cells"}} \phi(d_i/\epsilon, t_j) \\ +\; &\sum_{N_\epsilon \leq i \leq M_\epsilon} \xi_i d_i^2 \sum_{j \text{ over "no-shift cells"}} \psi(d_i/\epsilon, t_j). \end{aligned} \tag{19}$$

We again can replace the sum over the cells with an average value of ϕ and ψ respectively, and substitute asymptotic expressions to obtain

$$\sum_{\substack{N_\epsilon \leq i \leq M_\epsilon \\ \text{"shift" cell}}} \left[1 - \frac{2^{\frac{p+2}{p+1}}}{i}\right] \frac{2\pi p}{i^{2p+1}}\, \widehat{\phi}_i \;+\; \sum_{\substack{N_\epsilon \leq i \leq N_\epsilon \\ \text{"no shift" cell}}} \left[1 - \frac{2^{\frac{p+2}{p+1}}}{i}\right] \frac{2\pi p}{i^{2p+1}}\, \widehat{\psi}_i. \tag{20}$$

Now we wish to estimate this as an integral. We define $\mathcal{S}_\epsilon$ and $\mathcal{N}_\epsilon$ as subsets of $[N_\epsilon, M_\epsilon]$ given by

$$\mathcal{S}_\epsilon = \{[i, i+1] : \text{ the annulus between } S_i \text{ and } S_{i+1} \text{ has "shifts" }\}$$

and $\mathcal{N}_\epsilon = [N_\epsilon, M_\epsilon] \setminus \mathcal{S}_\epsilon$, so will also be union of disjoint intervals. Thus, asymptotically we can estimate the sums in equation (20) as the sum of the integrals

$$\int_{\mathcal{S}_\epsilon} \left[1 - \frac{2^{\frac{p+2}{p+1}}}{x}\right] \frac{2\pi p}{x^{2p+1}}\, \widehat{\phi}_x\; dx + \int_{\mathcal{N}_\epsilon} \left[1 - \frac{2^{\frac{p+2}{p+1}}}{x}\right] \frac{2\pi p}{x^{2p+1}}\, \widehat{\psi}_x\; dx.$$

Using the same change of variables as before, $y = (p/\epsilon)x^{-p-1} = d/\epsilon$, the two sets $\mathcal{S}_\epsilon, \mathcal{N}_\epsilon$ are transformed into two subsets of $\widehat{\mathcal{S}}_\epsilon, \widehat{\mathcal{N}}_\epsilon \subset [\sqrt{2}, 2]$. Again these subsets are unions of intervals and we obtain the sum of two integrals, each one as in (16), one over $\widehat{\mathcal{S}}_\epsilon$ and one over $\widehat{\mathcal{N}}_\epsilon$. We claim that as $\epsilon \to 0$, both of these integrals converge. The key is to understand what happens to $\widehat{\mathcal{S}}_\epsilon$ and $\widehat{\mathcal{N}}_\epsilon$ as $\epsilon \to 0$. $\mathcal{S}_\epsilon$ consists of approximately $2\pi(M_\epsilon - N_\epsilon)/p$ intervals each of unit length and evenly distributed in the interval $[N_\epsilon, M_\epsilon]$ and separated by a distance of $p/(2\pi)$. This means that $\widehat{\mathcal{S}}_\epsilon$ also consists of this number of disjoint intervals and also distributed throughout the interval $[\sqrt{2}, 2]$. As $\epsilon \to 0$, the set $\widehat{\mathcal{S}}_\epsilon$ becomes dense in $[\sqrt{2}, 2]$ (as does $\widehat{\mathcal{N}}_\epsilon$). Thus the integral of any continuous function g over $\widehat{\mathcal{S}}_\epsilon$ will limit to some integral of g over the entire interval $[\sqrt{2}, 2]$. The intervals of $\mathcal{S}_\epsilon$ are uniformly distributed in $[N_\epsilon, M_\epsilon]$ with density $2\pi/p$. This means that $\widehat{\mathcal{S}}_\epsilon$ will asymptotically be uniformly distributed in $[\sqrt{2}, 2]$ with density $2\pi/p$ as well.

Therefore, for any continuous $g : [\sqrt{2}, 2] \to \mathbb{R}$, we have that

$$\lim_{\epsilon \to 0} \int_{\widehat{\mathcal{S}}_\epsilon} g(y)\ dy = \frac{2\pi}{p} \int_{\sqrt{2}}^{2} g(y)\ dy,$$

and

$$\lim_{\epsilon \to 0} \int_{\widehat{\mathcal{N}}_\epsilon} g(y)\ dy = \left(1 - \frac{2\pi}{p}\right) \int_{\sqrt{2}}^{2} g(y)\ dy.$$

Thus, we have:

Lemma 4. *In the case $p > 2\pi$, the sum contribution of the "middle" range to the area of E_d^ϵ is asymptotic to*

$$C_p\, \frac{2\pi p^{\frac{2}{p+1}-1}}{p+1}\, \epsilon^{2-\frac{2}{p+1}},$$

where this time C_p represents the weighted sum of the two integrals, one for $\widehat{\Phi}$ and one for $\widehat{\Psi}$.

Finally, recall that, for clarity, our discussion was restricted to assuming that the annuli are rectangles and not actual annuli. What changes if instead of developing formulas for ϕ based on thin rectangles we develop similar formulas for ϕ based on the actual situation of annuli? Clearly the formulas themselves will be different, but only in the details, not in the general form. In fact, the overall situation changes very little. It is for this reason that we illustrated with rectangles rather than with annuli. The conclusion is still the same.

Thus, putting together lemmas 1, 2 and either lemma 3 or lemma 4, we get the following theorem.

Theorem 2. *Let $d \in (0, 2)$ and $C > 0$ be given. Then there is a countable set $E_d \subset \mathbb{R}^2$, of the form described above, so that the Minkowski dimension of E_d is d, the set E_d is Minkowski measurable, and the Minkowski content of E_d is exactly C.*

2.2. **N-dimensional case.** The same construction can easily be performed in $\mathbb{R}^N$ and the details of the estimation extended. For any $p > 0$, we set S_n to be a sphere of radius n^{-p} centered at 0 and we then choose points on S_n whose minimum separation is asymptotically $n^{-p} - (n+1)^{-p}$. Now, it is not so simple to choose points on a sphere in $\mathbb{R}^N$ which are all uniformly placed and uniformly spaced (usually it is impossible to do this). In fact, in general, it is a difficult computational problem to try to even approximately do this (see [7, 8]). However, we don't need to have the spacing completely uniform. All we really need is that the spacing is asymptotically within a constant multiple of $n^{-p} - (n+1)^{-p}$ and that we can do the same type of scaling that we did in the example in $\mathbb{R}^2$, above.

If the points were to be placed in $\mathbb{R}^{N-1}$ instead of on the sphere, we would place them on a scaled version of the integer lattice, where the points are spaced on hypercubes of side length $n^{-p}-(n+1)^{-p}$. This would result in a range of inter-point spacing from $n^{-p}-(n+1)^{-p}$ to $\sqrt{N-1}(n^{-p}-(n+1)^{-p})$ for the "diagonal" distance. As this structure scales nicely, similar arguments to those above would show convergence.

For our placement on the sphere of radius n^{-p} in $\mathbb{R}^N$, we use the "polar coordinates subdivision" method [8]. The procedure recurses down the dimensions. To describe this method, let x_i be the generalized spherical coordinates, that is

$$\begin{aligned} x_i &= \rho\left(\prod_{j=1}^{i-1}\cos(\phi_j)\right)\sin(\phi_i), \quad 1 \le i \le N-2, \quad \text{and} \\ x_{N-1} &= \rho\left(\prod_{j=1}^{N-2}\cos(\phi_j)\right)\cos(\phi_N - 1), \end{aligned}$$

with $\phi_i \in [0,\pi)$ for $i = 1,\ldots,N-2$ and $\phi_{N-1} \in [0,2\pi)$. We set $\rho = n^{-p}$ for the sphere S_n. With $\Delta = n^{-p}-(n+1)^{-p}$, the desired spacing between the points, we choose a grid of $\lfloor \pi n^{-p}/\Delta \rfloor$ equally spaced points in $[0,\pi)$ for the grid in the ϕ_1 direction. When we fix ϕ_1 to any of these values, we have an $N-2$ dimensional sphere whose radius is $n^{-p}\cos(\phi_1)$. We recurse by choosing a grid of $\lfloor \pi n^{-p}\cos(\phi_1)/\Delta \rfloor$ equally spaced points in $[0,\pi)$ for the grid in the ϕ_2 direction on this particular $N-2$ dimensional spherical slice.

Asymptotically, this results in

$$\frac{2\pi^{N/2} n^{-\frac{N-1}{p}}}{\Gamma(N/2)(n^{-p}-(n+1)^{-p})^{N-1}} = \frac{\text{Surface area of sphere of radius } n^{-p}}{\text{hypervolume of cell in square grid}}$$

points on the sphere of radius n^{-p}, which is just what one would expect for a "hypersquare" grid in $\mathbb{R}^{N-1}$ of side length $n^{-p}-(n+1)^{-p}$ in a region with total volume equal to the surface area of the given sphere.

The estimation of the volume of the "outer" and "inner" ranges are more-or-less exactly analogous as in $\mathbb{R}^2$, the only change being that the "inner" range doesn't start until there is complete overlap for the "diagonal" points on S_n and S_{n+1}. This at most multiplies M_ϵ by a constant factor of $N^{1/(2p+2)}$; the starting point of the "middle" range remains unchanged.

For the "middle" range, the estimate will involve the same reasoning as in $\mathbb{R}^2$, the formulas being much more involved but the logic remaining the same.

The Minkowski dimension of the resulting countable subset of $\mathbb{R}^N$ is $N/(p+1)$ and the content is positive and finite. With this, we can obtain the following:

Theorem 3. *Let $d \in (0,N)$ and $C > 0$ be given. Then there is a countable set $E_d \subset \mathbb{R}^N$, of the form described above, so that the Minkowski dimension of E_d is d, the set E_d is Minkowski measurable, and the Minkowski content of E_d is exactly C.*

2.3. **Alternative construction for dilations using the l^∞ norm.** If we are willing to modify the definition of the dilation of a set by using a slightly different distance, the above construction can be repeated and the analysis greatly simplified. For this section, we will use the definition

$$A^\epsilon = \{y : \|a-y\|_\infty < \epsilon, \text{ for some } a \in A\},$$

so that we use the supremum norm rather than the usual Euclidean distance. We again obtain the same result, that for any $d \in (0,N)$ and any $C > 0$ there is a set

$E_d \subset \mathbb{R}^N$ with Minkowski dimension d and with Minkowski content exactly equal to C.

It is not hard to see that computing the Minkowski dimension with either definition of dilation will yeild the same result. However, it is not clear if both dilations give the same class of Minkowski measurable sets – it is likely that they do not.

This time the construction uses concentric hypercubes rather than concentric spheres (see Fig. 6). The hypercube S_n is centered at the origin and has side length $2n^{-p}$, just as in the previous construction. We again place finitely many points on S_n. For each $N-1$ dimensional face, we place the points along coordinate directions with a separation of $n^{-p-1}(\lfloor 1/p \rfloor + 1)$ (which is asymptotic to $n^{-p} - (n+1)^{-p}$). This gives, asymptotically, a total of $(2n(\lfloor 1/p \rfloor + 1) + 1)^{n-1}$ points on each face.

The major simplification is that, with the new definition of an ϵ-dilation of a set, the geometry is much simpler. For instance, once the ϵ-dilation of points on S_n intersect, the union of these ϵ-squares will form a rectangular "tubular" neighborhood of the entire square S_n and this area is simple to compute. Furthermore, either the tubular neighborhood of S_n will join with that of S_{n+1}, completely filling the region inbetween, or they don't overlap at all. Thus we don't have the complication of all the different types of overlap.

It is also clearly much simpler to arrange points on the faces of these squares than on the sphere, and this substantially simplifies the analysis.

We again use Lebesgue measure to measure the volume of E_d^ϵ (with the new definition of ϵ dilation). We obtain the following result:

Theorem 4. *Let $E \subset \mathbb{R}^N$ be the countable subset described above. Then using the l^∞ ϵ-dilation, we have that the Minkowski dimension of E is $N/(p+1)$ and the Minkowski content is*

$$2^{\frac{2Np+N}{p+1}} \left(a^{\frac{Np-p-1}{p+1}} p^{\frac{-Np}{p+1}} \right) + \frac{2^{\frac{2Np+N}{p+1}} N (p^{\frac{p-Np+1}{p+1}} - a^{\frac{p-Np+1}{p+1}})}{p - Np + 1},$$

where $a = \lfloor 1/p \rfloor + 1$.

Notice that we have the same Minkowski dimension as the set constructed with spheres. This is not a surprise as it is possible to show that there is a bi-Lipschitz mapping from one set onto the other, so their dimensions must agree.

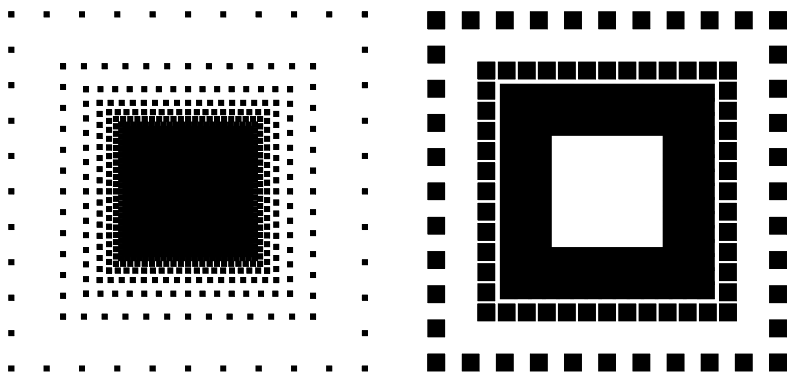

FIGURE 6. Illustration of squares and "square" dilations.

3. A strictly increasing C^1 function which does not preserve Minkowski measurability

Clearly any linear function $f : \mathbb{R} \to \mathbb{R}$ will preserve Minkowski measurability and simply scale the Minkowski content. It is also easy to see that any bi-Lipschitz function will preserve the Minkowski dimension. However, it is not so clear what are necessary and sufficient conditions on a function f to preserve Minkowski measurability. The following example is quite surprising in that continuity in the local distortion is not sufficient to guarantee that a function preserves Minkowski measurability. This means that a function has to satisfy quite strong conditions in order to guarantee that it preserves Minkowski measurability. Contrast this to the case of Hausdorff measures, where if $0 < \mathcal{H}^d(A) < \infty$ and f is bi-Lipschitz then $0 < \mathcal{H}^d(f(A)) < \infty$.

We first start with a lemma which describes a general construction. Since the Minkowski dimension, measurability, and content (if it exists) of a compact subset of $\mathbb{R}$ depends only on the sequence of "gap lengths", this construction starts with these gap lengths and shows how to construct a whole family of compact sets with exactly these "gap lengths". The idea is similar to that given in [9, 10].

If $\prec$ is a total order on a set S, a *cut* is a partition $\{L, R\}$ with $L \cup R = S$ and for all $l \in L$ and $r \in R$ we have $l \prec r$. Note that we allow either L or R to be empty.

Lemma 5. *Let a_n be a positive summable sequence. Let $\prec$ be any total order on $\mathbb{N}$. Define the set*

$$C_a = \{\sum_{i \in S_1} a_i : (S_1, S_2) \text{ a cut of } \mathbb{N} \text{ in the order } \prec\}.$$

Then C_a is a closed subset of $[0, \sum_n a_n]$ with Lebesgue measure zero.

Proof. Let $I = [0, \sum_n a_n]$.

The two points $\bar{x}_n = a_n + \sum_{i \prec n} a_i$ and $\underline{x}_n = \sum_{i \prec n} a_i$ are both in C_a. By the definition of C_a, there are no points $y \in C_a$ with $\underline{x}_n < y < \bar{x}_n$. Since $\bar{x}_n - \underline{x}_n = a_n$, we see that $I \setminus C_a$ contains an open interval of length a_n, namely the interval $(\underline{x}_n, \bar{x}_n)$. Since this is true for each n and all these intervals are disjoint, $|I \setminus C_a| \geq \sum_n a_n = |I|$ and so $|C_a| = 0$.

To show that C_a is closed, suppose that $x_n \nearrow x$ where $x_n \in C_a$. Let (S_1^n, S_2^n) be the cut of $\mathbb{N}$ associated with x_n and define $S_1 = \cap_n S_1^n$ and $z = \sum_{i \in S_1} a_i$. We will show that $x = z$. First, by definition we see that $x_n \leq z$ for all n and so $x \leq z$. Let $\epsilon > 0$ be given and choose N so that $n \geq N$ implies that $\sum_{i \geq n} a_i < \epsilon$. Now, there is some M so that for any $n \geq M$ we have $S_1 \setminus S_1^n \subset \{i \geq N\}$ so for such n we have $\sum_{i \in S_1 \setminus S_1^n} a_i < \epsilon$. However, this means that $|z - x_n| < \epsilon$. The proof for $x_n \searrow x$ is similar.

Since any convergent sequence $x_n \to x$ contains a monotone subsequence, we are done. □

We now describe our construction of the promised function.

Let $a_n = 1/3 \times 1/n^s$ where $s = \log(3)/\log(2)$. Then we can also write $a_n = 3^{-1-\log(n)/\log(2)}$. We also consider $b_n = 3^{-1-\lfloor \log(n)/\log(2) \rfloor}$ so that $b_n = 1/3, 1/9, 1/9, 1/27, 1/27, 1/27, \ldots$. That is, b_n is the set of gap lengths of the standard middle-1/3 Cantor set.

From Theorem 1 we see that any closed set with complementary gap lengths a_n is Minkowski measurable and any closed set with complementary gap lengths b_n is not Minkowski measurable. The nonmeasurability of the classical Cantor set was noticed in [5] and is a consequence of the "oscillations" of $V(\epsilon)$. This can be seen explicitly in Eq. (1.11) of [3], for example. We construct a C^1 function which takes C_a bijectively onto C_b.

We see that

$$c_n := \frac{b_n}{a_n} = 3^{\log(n)/\log(2) - \lfloor \log(n)/\log(2) \rfloor} = 3^{\mathrm{frac}(\log(n)/\log(2))} \tag{21}$$

satisfies

$$1 \le c_n \le 3.$$

First we note that $\{c_n\}$ is dense in $[1, 3]$. To see this, for $n = 2^k + i$ with $0 \le i < 2^k$ we have

$$\log(n)/\log(2) = \log_2(2^k + i) = k + \log_2(1 + i/2^k).$$

Since numbers of the form $1 + i/2^k$ are dense in $[1, 2]$, $\log_2(1 + i/2^k)$ is dense in $[0, 1]$ and thus c_n is dense in $[1, 3]$.

Define the total order $\prec$ on $\mathbb{N}$ by $n \prec m$ if either $c_n < c_m$ or if $c_n = c_m$ but $n < m$. Using this order, we define C_a and C_b as in the lemma. Further, we use the notation $I_a = [0, \sum_n a_n]$ and similarly for I_b. Notice that if $(\underline{x}_n, \bar{x}_n)$ and $(\underline{x}_m, \bar{x}_m)$ are two complementary intervals for C_a with $\bar{x}_n < \underline{x}_m$ then $c_n \le c_m$. This fact is important.

We define the function $f : I_a \to I_b$ by first defining $f : C_a \to C_b$ and extending. For $x \in C_a$, we see that there is some cut (S_1, S_2) of $\mathbb{N}$ with $x = \sum_{i \in S_1} a_i$. So, using this we define

$$f(x) = \sum_{i \in S_1} b_i. \tag{22}$$

Now we extend f to I_a by linearly extending it to each complementary interval. That is, for $t \in (\underline{x}_n, \bar{x}_n)$ we define

$$f(t) = \frac{f(\bar{x}_n) - f(\underline{x}_n)}{\bar{x}_n - \underline{x}_n}(t - \underline{x}_n) + f(\underline{x}_n) = c_n(t - \underline{x}_n) + f(\underline{x}_n) = c_n(t - \underline{x}_n) + \sum_{i \prec n} b_i.$$

Notice that the slope of f on $(\underline{x}_n, \bar{x}_n)$ is c_n.

Proposition 1. *The function f is continuously differentiable.*

Proof. We show this by explicitly exhibiting the derivative of f, which we will call g.

First we define g on $I_a \setminus C_a$ by $g(x) = c_n$ for any $x \in (\underline{x}_n, \bar{x}_n)$. We claim that we can continuously extend g to all of I_a. To do this, we define g on C_a in such a way to make it continuous. For $x = \sum_{i \in S_1} a_i$, where (S_1, S_2) is the defining cut for $x \in C_a$, we let

$$g(x) = \sup\{c_n : n \in S_1\}. \tag{23}$$

Notice that g is increasing on C_a and constant on each component of $I_a \setminus C_a$. In fact, g is very much analogous to the Cantor ternary function.

Let $x \in C_a$ be defined by the cut (S_1, S_2). Now, suppose that $y_{n_k} \in (\underline{x}_{n_k}, \bar{x}_{n_k})$ with $y_{n_k} \nearrow x$. This means that $n_k \in S_1$ and thus $c_{n_k} \le g(x)$. Suppose that there is some d with $c_{n_k} \le d < g(x)$. As $\{c_i\}$ is dense in $[1, 3]$, there is an M with $c_{n_k} < c_M < g(x)$ which implies that $y_{n_k} \le \underline{x}_M < \bar{x}_M \le x$ and so $x - y_{n_k} \ge a_M > 0$, a contradiction. Thus $c_{n_k} \nearrow g(x)$. In a similar way we can see that if $y_{n_k} \searrow x$ with $y_{n_k} \in (\underline{x}_{n_k}, \bar{x}_{n_k})$ then $c_{n_k} \searrow g(x)$.

To finish the proof that g is continuous, we just note that if $y \in C_a$ with $y = \sum_{i \in S_1} a_i$, then for any $n \in S_1$ and $m \in S_2$, we have $c_n \le g(y) \le c_m$. This implies that it is enough to only consider sequences from the gaps which converge to x.

Next we show that $f'(x) = g(x)$ for all x in the interior of I_a. If $x \in I_a \setminus C_a$, then $x \in (\underline{x}_n, \bar{x}_n)$ for some n and thus $x = \underline{x}_n + (x - \underline{x}_n)$ so

$$f(x) = \sum_{m \prec n} b_m + c_n(x - \underline{x}_n) = \sum_{n \prec m} a_n c_m + c_n(x - \underline{x}_n) = \sum_{m \prec n} \int_{\underline{x}_m}^{\bar{x}_m} g(t)\, dt + \int_{\underline{x}_n}^{x} g(t)\, dt = \int_0^x g(t)\, dt.$$

On the other hand, if $x = \sum_{i \in S_1} a_i$, then we have

$$f(x) = \sum_{i \in S_i} b_i = \sum_{i \in S_1} a_i c_i = \sum_{i \in S_1} \int_{\underline{x}_i}^{\bar{x}_i} g(t)\ dt = \int_0^x g(t)\ dt.$$

□

Notice that $1 \leq g(t) \leq 3$ for all t, and thus $1 \leq f'(t) \leq 3$ for all $t \in I_a$.

4. Closing comments and further questions

The situation in one dimension is very nicely characterized by the previous work in [5, 6]. It would be interesting to have such a characterization for higher dimensions (even two dimensions). However, it is likely that such a characterization is not possible or at least not such a nice characterization. Perhaps it might be possible for countable sets. In our construction, we use the simplest "type" of countable compact set in that our sets all have only one limit point. This means that all other points have only finitely many nearest neighbours, which simplifies the situation. A characterization of the Minkowski dimension and measurability of an arbitrary countable compact set would have to contend with perhaps infinitely many limit points. There is a nice inductive characterization of countable compact sets given in a classic paper by Mazurkeiwicz and Sierpinski [11]. One approach might be based on a transfinite induction on the type of the countable compact set.

A further direction to explore is to determine the precise properties that a function f must satisfy in order to preserve Minkowski measurability. This is not a simple task even for subsets of $\mathbb{R}$, as our example indicates. One would think that having the "local distortion" change in a continuous fashion would be enough, but our example shows that it is not a strong enough condition.

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