

Dimensions of linear Cantor sets

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Abstract This paper provides a survey of some results on the size of Cantor-type subsets of $\mathbb{R}$. More specifically, we use various metric dimensions to measure the size. These include the Hausdorff, Packing, Box and Assouad dimensions. We also provide an introduction to the newer and more general Φ -dimensions, which allow fine-grained distinctions of the local growth rate of the size of coverings by intervals.

We concentrate on results that have grown out of our previous collaborations with Carlos Cabrelli, a great mathematician and friend to many. We enthusiastically dedicate this paper to him in appreciation of the support and encouragement he has given to us.

1 Introduction

A compact subset E of $\mathbb{R}$ is completely determined by its set of *complementary intervals* (or *gaps*). However, it might be surprising to learn that interesting geometric properties can be determined knowing only the lengths of these intervals.

We will primarily be interested in the values of various metric dimensions of E . If the Lebesgue measure of E is positive, its Hausdorff dimension is maximal at one, and thus many of these dimensions will also be maximal at one. Thus we will assume that the Lebesgue measure of E is zero. Furthermore, with no real loss of generality we will assume that $E \subset [0, 1]$ with $0, 1 \in E$, and so the bounded gaps I_n satisfy $[0, 1] \setminus E = \bigcup_n I_n$. To avoid annoying and trivial boundary cases, we assume

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that E is a perfect set and thus there are infinitely many such gaps. Throughout we normally use the notation $a_n = |I_n|$ for the sequence of gap lengths. Notice that our assumptions on E imply that $\sum_n a_n = 1$.

There are infinitely many different compact E which have the same sequence of gap lengths. However, given a non-increasing sequence $a = (a_n)$ with $a_n > 0$ and $\sum_n a_n = 1$, there is a natural way to construct a perfect Cantor-type set C_a . We start by removing an open interval I_1 of length a_1 from $[0, 1]$ whose position ends up being uniquely specified by the requirement that the following process can continue indefinitely. In the next stage, we remove an open interval I_2 of length a_2 from the left remaining closed interval and also remove an open interval I_3 of length a_3 from the right remaining closed interval. In the k th stage we remove open intervals of lengths $a_{2^k}, a_{2^k+1}, \dots, a_{2^{k+1}-1}$ in an “interleaving” fashion. The first image in Fig. 1 illustrates the placement of the gaps in this construction.

The position of the left endpoint of the first removed gap (the one with length a_1) is given by the sum of all the gap lengths which are later removed to its “left”. Because $\sum_n a_n = 1$, this is the only possible location if the process is to continue indefinitely; the second image in Fig. 1 illustrates this fact. In fact, for any $x \in C_a$, we have

$$x = \sum_{I_i < x} a_i, \quad (1)$$

that is, the location of every point in C_a is given by the sum of the gaps “to its left”. We use the notation $I < x$ for an interval I and point x to mean that $y < x$ for all $y \in I$.

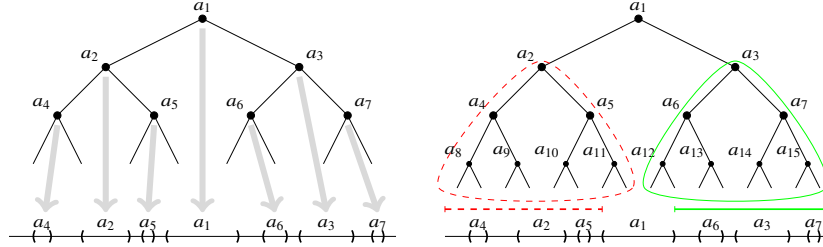


Fig. 1 The placement of gaps and unique location for the placement in constructing C_a .

Of course, even if the sequence (a_n) is not decreasing we can still complete the same construction and obtain a perfect and compact set C_a where the complementary gaps will have lengths corresponding to terms of the sequence (a_n) . It is also possible to obtain many other compact sets $F \subset [0, 1]$ with gap lengths given by (a_n) , even countable sets. There is a literature and much to learn about these *rearrangements*, but we will focus on the set C_a when $a_n \searrow 0$.

When $a_{2^k} = a_{2^k+1} = \dots = a_{2^{k+1}-1}$ for all k the set C_a is called a *central Cantor set* and has nice symmetry properties. The classical middle 1/3 Cantor set is such an example.

Suppose we start with two such sequences (a_n) and (b_n) with constant $C > 0$ so that $b_n/a_n \leq C$ for all n . Then the natural increasing homeomorphism $f : C_a \rightarrow C_b$ given by (see Cabrelli et al. [4])

$$f(x) = \sum_{I_i < x} b_i, \text{ (where } \{I_i\} \text{ are the gaps from } C_a) \quad (2)$$

satisfies $|f(x) - f(y)| \leq C|x - y|$ and so is Lipschitz. This observation motivates our next definition.

Notation 1 For two functions f, g we say that $f \preceq g$ if there is a $c > 0$ so that $f(x) \leq cg(x)$. Furthermore, we say $f \asymp g$ if $f \preceq g$ and $g \preceq f$.

For sequences this means $(a_n) \preceq (b_n)$ when $a_n \leq cb_n$ for $c > 0$ and $(a_n) \asymp (b_n)$ when $c_1 \leq a_n/b_n \leq c_2$ for all n .

If $a_n \asymp b_n$, then the map f in (2) is bi-Lipschitz. This will generally mean that C_a and C_b have the same dimension.

A positive non-increasing sequence (a_n) is *doubling* if there is a $\kappa > 0$ so that $a_n \leq \kappa a_{2n}$ for all n . In this case, (a_n) is equivalent to the sequence of a central Cantor set and so C_a is bi-Lipschitz equivalent to a central Cantor set. To see this, let $b_n = a_{2^k}$ for $2^k \leq n < 2^{k+1}$. We have $1/\kappa \leq a_n/a_{2^k} \leq 1$ when $2^k \leq n < 2^{k+1}$ and so $b_n \asymp a_n$. For instance, the classical middle 1/3 Cantor set is bi-Lipschitz equivalent to C_a with $a_n = n^{-\log 3/\log 2} = 3^{-\log n/\log 2}$.

2 Hausdorff and Packing dimensions of Cantor sets

In their seminal paper [2], Besicovitch and Taylor investigated the relationship between the gap lengths and the Hausdorff dimension and measure of the set. Among the many results on the dimension and measure of general rearrangements, we highlight three specific ones that are closest to our theme.

First, however, we give the reader a brief reminder of the definition of the Hausdorff measure and dimension. We give these definitions for subsets of $\mathbb{R}$ but they are also defined in all metric spaces, the books [6, 8, 11] are good references for dimensions in metric spaces. We begin by recalling that a δ -covering is a countable family $U_i \subset \mathbb{R}$ such that $|U_i| < \delta$ (by $|S|$ we mean the diameter of the set S). Then for a given $s > 0$ and $E \subset \mathbb{R}$, we define

$$\mathcal{H}_\delta^s(E) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-covering of } E \right\} \quad (3)$$

and finally $\mathcal{H}^s(E) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^s(E)$. The measure constructed this way is called the *s-dimensional Hausdorff measure*, is Borel regular and is a constant multiple of Lebesgue measure whenever $s \in \mathbb{N}$. It is not hard to show that if $0 < s < t$ and $\mathcal{H}^s(E) < \infty$ then $\mathcal{H}^t(E) = 0$, which also means that if $\mathcal{H}^t(E) > 0$ then

$\mathcal{H}^s(E) = \infty$. Because of this, for Borel $E \subset \mathbb{R}$, there is a unique value d for which $\mathcal{H}^s(E) = \infty$ when $t < d$ and $\mathcal{H}^s(E) = 0$ when $t > d$; the value of $\mathcal{H}^d(E)$ could, in general, be anything in $[0, \infty]$. This special cutoff is the *Hausdorff dimension* (see Fig. 2)

$$\dim_H E = \inf\{s > 0 : \mathcal{H}^s(E) = 0\} = \sup\{s > 0 : \mathcal{H}^s(E) = \infty\}.$$

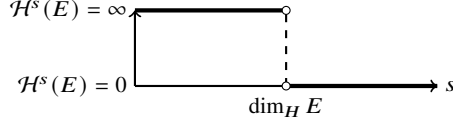


Fig. 2 The cutoff behaviour in $\mathcal{H}^s(E)$

Proposition 1 ([2]) *Let (a_n) be a non-increasing, summable and positive sequence. Then*

1. $\dim_H C_a = \liminf \frac{\log m}{-\log \frac{1}{m} \sum_{j \geq m} a_j} = \inf\{s > 0 : \liminf n \left(\frac{1}{n} \sum_{i \geq n} a_i\right)^s < \infty\}.$
2. $\dim_H C_a$ is maximal among all sets with gap sequence (a_n) .
3. $\frac{1}{4} \liminf n \left(\frac{1}{n} \sum_{i \geq n} a_i\right)^s \leq \mathcal{H}^s(C_a) \leq \liminf n \left(\frac{1}{n} \sum_{i \geq n} a_i\right)^s.$

The tail sums, $\sum_{i \geq n} a_i$, appear in these formulas for a very natural geometric reason, as this is the combined length of the n intervals which remain after removing the first $n-1$ open intervals. When the gaps are removed as above with a_n decreasing, the lengths of the remaining closed intervals are all “almost” their average and so form a nice cover. More specifically, if we let $a'_n = a_{2^{k-1}}$ for $2^{k-1} \leq n < 2^k$, Besicovitch and Taylor show that for $0 < s \leq 1$

$$\frac{1}{2} 2n \left(\frac{1}{2n} \sum_{i \geq 2n} a_i \right)^s \leq n \left(\frac{1}{n} \sum_{i \geq n} a'_i \right)^s \leq n \left(\frac{1}{n} \sum_{i \geq n} a_i \right)^s,$$

and so C_a is always “almost” a central Cantor set (at least from the standpoint of the Hausdorff dimension and measure). As a concrete example, when $a_n = n^{-p}$ with $p > 1$, in Cabrelli et al. [5] it is shown that

$$\frac{1}{8} \left(\frac{2^p}{2^p - 2} \right)^{1/p} \leq \mathcal{H}^{1/p}(C_a) \leq \left(\frac{1}{p-1} \right)^{1/p}.$$

The papers [4, 13] refine and extend these results to more general Hausdorff measures, so we will first need to discuss these. For our purposes, a *gauge function* (also called a dimension function) is a function $h : (0, c) \rightarrow (0, \infty]$, for some $c > 0$, which is continuous, non-decreasing and with $h(x) \rightarrow 0$ as $x \rightarrow 0$. Such a function

is *doubling* if there is some $\tau > 0$ so that $h(x) \geq \tau h(2x)$ for all x in the domain of h . We will denote the set of all doubling gauge functions by $\mathcal{G}$. We mention that only some of our results require the doubling condition, but we make the blanket assumption for clarity. The interested reader is invited to check the references for precisely what condition is necessary for each result. Any of the power functions $h_s(x) = x^s$, $s > 0$, is a standard example of a doubling gauge function.

Given $h \in \mathcal{G}$, the h -Hausdorff measure of a set E , $\mathcal{H}^h(E)$, is defined via the same process as $\mathcal{H}^s$:

$$\mathcal{H}_\delta^h(E) = \inf \left\{ \sum_{i=1}^{\infty} h(|U_i|) : \{U_i\} \text{ is a } \delta\text{-covering of } E \right\}$$

and $\mathcal{H}^h(E) = \lim_{\delta \rightarrow 0} \mathcal{H}_\delta^h(E)$. As before, this results in a regular Borel measure on $\mathbb{R}$, where this time the “size” of an interval is measured by $h(|I|)$ rather than by $|I|^s$.

If $(a_n) \preceq (b_n)$ then $\mathcal{H}^h(C_a) \leq C \cdot \mathcal{H}^h(C_b)$ for some constant $C > 0$ and thus if $(a_n) \asymp (b_n)$ then $\mathcal{H}^h(C_a)$ and $\mathcal{H}^h(C_b)$ are zero, finite and positive, or infinite for the same gauge functions h .

In [4] it is also shown that for any decreasing (a_n) there is a concave $h_a \in \mathcal{G}$ for which

$$1/32 \leq \mathcal{H}^{h_a}(C_a) \leq 1$$

and thus C_a is always *dimensional* for some $h \in \mathcal{G}$. An example of such an h_a is as follows. First, given (a_n) , we define $b_n = \frac{1}{n} \sum_{i \geq n} a_i$, the averaged tail sum. Then define $f : [1, \infty) \rightarrow \mathbb{R}$ by $f(x) = b_k(k+1-x) + b_{k+1}(x-k)$ for $x \in [k, k+1)$. Finally, we let

$$h_a(t) = \begin{cases} \frac{1}{f^{-1}(t)}, & \text{if } 0 < t \leq b_1; \\ h(0) = 0. \end{cases}$$

This choice results in $h_a(b_k) = 1/k$, which is the key property.

As a natural extension of these ideas, in [13] the authors prove that for any $h \in \mathcal{G}$,

$$\frac{1}{4} \liminf n \cdot h\left(\frac{1}{n} \sum_{i \geq n} a_i\right) \leq \mathcal{H}^h(C_a) \leq 4 \liminf n \cdot h\left(\frac{1}{n} \sum_{i \geq n} a_i\right), \quad (4)$$

which very nicely characterizes the behaviour of $\mathcal{H}^h(C_a)$. Note that in the example above this limit inferior is equal to one.

The authors in [13] also prove similar results about the h -packing measure, so again we will first remind the readers about this. Again we give these definitions for subsets of $\mathbb{R}$ but they are defined in all metric spaces. A δ -*packing* of a non-empty set $E \subset \mathbb{R}$ is a disjoint family of open intervals with centers in E and with lengths all less than δ . Given this, the h -*packing pre-measure* $\mathcal{P}_0^h$ is defined by

$$\mathcal{P}_0^h(E) = \lim_{\delta \rightarrow 0} \sup \left\{ \sum_{i=1}^{\infty} h(|U_i|) : \{U_i\} \text{ is a } \delta\text{-packing of } E \right\}. \quad (5)$$

Unfortunately it turns out, for technical reasons, that $\mathcal{P}_0^h$ is not a measure and thus in order to produce a measure we need one additional step:

$$\mathcal{P}^h(E) = \inf \left\{ \sum_{i=1}^{\infty} \mathcal{P}_0^h(U_i) : E \subseteq \bigcup_i U_i \right\}. \quad (6)$$

Because we assume that h is doubling, we have $\mathcal{H}^h(E) \leq \mathcal{P}^h(E) \leq \mathcal{P}_0^h(E)$.

The *pre-packing dimension* is the critical value for the packing pre-measure and is given by

$$\dim_{P_0} E = \sup \{s > 0 : \mathcal{P}_0^s(E) = \infty\} = \inf \{s > 0 : \mathcal{P}_0^s(E) = 0\}$$

while the *packing dimension* is the similar critical value for the packing measure $\mathcal{P}^s(E)$. The pre-packing dimension is known to coincide with the *upper box dimension*, which we will discuss shortly. Since $\mathcal{H}^s(E) \leq \mathcal{P}^s(E) \leq \mathcal{P}_0^s(E)$, we have $\dim_H E \leq \dim_P E \leq \dim_{P_0} E$. In fact, in [9] it is shown that for any compact $K \subset \mathbb{R}^n$, if $\mathcal{P}_0^s(K) < \infty$ then $\mathcal{P}^s(K) = \mathcal{P}_0^s(K)$.

As a dual to (4), the authors in [13] also show that

$$\frac{1}{8} \limsup n \cdot h\left(\frac{1}{n} \sum_{i \geq n} a_i\right) \leq \mathcal{P}_0^h(C_a) \leq 8 \limsup n \cdot h\left(\frac{1}{n} \sum_{i \geq n} a_i\right). \quad (7)$$

This means that C_a is simultaneously an h_a -packing and h_a -Hausdorff set (having positive and finite measure for both). Furthermore, the authors show that $h_a \asymp x^s$ if and only if $0 < \mathcal{H}^s(C_a) \leq \mathcal{P}_0^s(C_a) < \infty$. This naturally leads to asking if a similar relation holds for general gauge functions in $\mathcal{G}$, which we discuss in the next section.

3 Classifying C_a by its dimension partition

The inequalities (4) and (7) indicate that there is a strong relationship between the dimensional behaviour of C_a and the asymptotics of $h \in \mathcal{G}$. The authors in Cabrelli et al. [3] explore this by considering the way that C_a induces a partition on $\mathcal{G}$.

By the *dimension partition* of a set $C_a \subset \mathbb{R}$, we mean a partition of $\mathcal{G}$ into (six) sets $\mathcal{H}_\alpha^a \cap \mathcal{P}_\beta^a$, for $\alpha \leq \beta \in \{0, 1, \infty\}$, where

$$\mathcal{H}_\alpha^a = \{h \in \mathcal{G} : \mathcal{H}^h(C_a) = \alpha\} \text{ for } \alpha = 0, \infty,$$

$$\mathcal{H}_1^a = \{h \in \mathcal{G} : 0 < \mathcal{H}^h(C_a) < \infty\},$$

and $\mathcal{P}_\beta^a$ is defined similarly, but replacing $\mathcal{H}^h$ with $\mathcal{P}^h$.

Furthermore, we say that any $h \in \mathcal{H}_1^a \cap \mathcal{P}_1^a$ is *regular*, in which case we also say that C_a is *h-regular*.

First we note that, by (4), $\mathcal{H}_\alpha^a$ could be defined in terms of $\liminf n \cdot h(\frac{1}{n} \sum_{i \geq n} a_i)$ instead of $\mathcal{H}^a(C_a)$. The situation for $\mathcal{P}_\beta^a$ is more complicated since (7) involves the

	$\mathcal{P}_0^a$	$\mathcal{P}_1^a$	$\mathcal{P}_\infty^a$
$\mathcal{H}_0^a$	$\mathcal{H}^h(C_a) = 0$ $\mathcal{P}^h(C_a) = 0$	$\mathcal{H}^h(C_a) = 0$ h -packing set	$\mathcal{H}^h(C_a) = 0$ $\mathcal{P}^h(C_a) = \infty$
$\mathcal{H}_1^a$		h -regular set	h -Hausdorff set $\mathcal{P}^h(C_a) = \infty$
$\mathcal{H}_\infty^a$			$\mathcal{H}^h(C_a) = \infty$ $\mathcal{P}^h(C_a) = \infty$

Fig. 3 Classification of functions in $\mathcal{G}$ for C_a

packing pre-measure and not the packing measure. In general, $\mathcal{P}^h(E) \leq \mathcal{P}_0^h(E)$ but it is possible for $\mathcal{P}^h(E) = 0$ and $\mathcal{P}_0^h(E) > 0$, the countable set $\{0\} \cup \{1/n : n \in \mathbb{N}\}$ being an example. However, in [3] it is shown that the situation for C_a is simpler.

Proposition 2 ([3]) *For any $h \in \mathcal{G}$, $\mathcal{P}_0^h(C_a) = 0$ (or ∞) if and only if $\mathcal{P}^h(C_a) = 0$ (resp. ∞) and $0 < \mathcal{P}_0^h(C_a) < \infty$ if and only if $0 < \mathcal{P}^h(C_a) < \infty$. Thus $\dim_P C_a = \dim_{P_0} C_a$.*

This result, along with (4) and (7), mean that whenever $h_a \asymp h_b$ we also know that $\dim_H C_a = \dim_H C_b$ and $\dim_P C_a = \dim_P C_b$. However, it is possible for C_a and C_b to have the same Hausdorff and packing dimensions but not the same dimension partition, see Example 4.6 from [13].

Another consequence is that $\mathcal{P}_\beta^a$ could also be defined in terms of $\limsup n \cdot h(\frac{1}{n} \sum_{i \geq n} a_i)$. Furthermore, $h \in \mathcal{H}_1^a \cap \mathcal{P}_1^a$ if and only if $h(\frac{1}{n} \sum_{i \geq n} a_i) \asymp 1/n$, and so any two functions in $\mathcal{H}_1^a \cap \mathcal{P}_1^a$ are equivalent.

We say two sequences (a_n) and (b_n) are *tail-equivalent* if the sequence of tail sums are equivalent, i.e., if there are $c_1, c_2 > 0$ so that

$$c_1 \sum_{i \geq n} a_i \leq \sum_{i \geq n} b_i \leq c_2 \sum_{i \geq n} a_i \text{ for all } n.$$

We say two sequences (a_n) and (b_n) are *weak tail-equivalent* if there are $j, k \in \mathbb{N}$ so that

$$\sum_{i \geq n} a_i \geq \frac{1}{j} \sum_{i \geq jn} b_i \text{ and } \sum_{i \geq n} b_i \geq \frac{1}{k} \sum_{i \geq kn} a_i.$$

Notice that tail-equivalent implies weak tail-equivalent. The converse is not true, as can be seen by the example of $a_n = e^{-n}$ and $b_n = e^{-2n}$, which are weak tail-equivalent but not tail-equivalent.

The next result is a full characterization of when two Cantor sets C_a and C_b have the same dimension partitions, both in terms of the sets themselves and also in terms of their underlying sequences. In particular, notice that knowing that $\mathcal{H}_1^a \cap \mathcal{P}_1^a = \mathcal{H}_1^b \cap \mathcal{P}_1^b$ implies that all the other sets in the partition are also the same.

Proposition 3 ([3]) *The following are equivalent:*

1. $h_a \asymp h_b$;
2. C_a and C_b are h -regular for the same $h \in \mathcal{G}$;
3. $\mathcal{H}_\alpha^a \cap \mathcal{P}_\beta^a = \mathcal{H}_\alpha^b \cap \mathcal{P}_\beta^b$ for all $\alpha \leq \beta \in \{0, 1, \infty\}$;
4. (a_n) is weak tail-equivalent to (b_n) .

In some situations, weak tail-equivalent sequences are also tail-equivalent. The following result gives some equivalent conditions for this to occur.

Proposition 4 ([3]) *Let $\mathcal{W}_a = \{(b_n) : (b_n) \text{ is weak tail-equivalent to } (a_n)\}$. Every $(b_n) \in \mathcal{W}_a$ is tail-equivalent to (a_n) if and only if there is a $(b_n) \in \mathcal{W}_a$ so that $h_b^{-1}(x)$ is a doubling function.*

Thus we know that if h_a^{-1} is doubling, then weak tail-equivalence implies tail-equivalence and hence we can replace weak tail-equivalence in Proposition 3 with tail-equivalence.

Corollary 1 ([3]) *Suppose that h_a^{-1} is a doubling function. Then the dimension partitions of C_a and C_b are the same if and only if (a_n) and (b_n) are tail-equivalent.*

4 Assouad dimension

Another family of metric dimensions are those which are related to the *Assouad dimension*. The Assouad dimension was used by Assouad in the study of embeddings of a metric space into $\mathbb{R}^d$, see [11] for some historical discussion. There are many variations on the Assouad dimension which we will discuss and all of them are related to the *box dimensions*, so we will start with these. Since we are concentrating on subsets in $\mathbb{R}$ we will specialize to this setting, but these dimensions easily work in any metric space.

Given $E \subset \mathbb{R}$ and $r > 0$, we let $N_r(E)$ be the minimal number of balls (intervals) of radius r which cover E . The *upper box* and *lower box* dimensions are then defined as

$$\overline{\dim}_B E = \limsup_{r \rightarrow 0} \frac{\log N_r(E)}{-\log r} \quad \text{and} \quad \underline{\dim}_B E = \liminf_{r \rightarrow 0} \frac{\log N_r(E)}{-\log r}. \quad (8)$$

When $\overline{\dim}_B E = \underline{\dim}_B E$ we call the common value the *box dimension* and denote it by $\dim_B E$. For any non-empty and bounded E we have (see [8])

$$\dim_H E \leq \dim_P E \leq \overline{\dim}_B E \quad \text{and} \quad \dim_H E \leq \underline{\dim}_B E \leq \overline{\dim}_B E;$$

there are examples where all of the inequalities are strict. For our particular situation, given a non-increasing sequence (a_n) it turns out that (see [13] for more discussion)

$$\overline{\dim}_B C_a = \limsup \frac{\log m}{-\log a_m} = \dim_P C_a$$

and

$$\underline{\dim}_B C_a = \liminf \frac{\log m}{-\log \frac{1}{m} \sum_{j \geq m} a_j} = \dim_H C_a.$$

For these it is very important that C_a is “almost” a central Cantor set.

The Assouad dimensions use the same idea as the box dimensions to measure size, but now “localizes” by taking two different scales $0 < r < R$ and covering $E \cap B(x, R)$ with balls of size r (see Fig. 4). The (upper) Assouad dimension is given as

$$\dim_A(E) = \inf \left\{ \alpha > 0 : \exists C, \rho > 0, \forall 0 < r < R \leq \rho, \sup_{x \in E} N_r(B(x, R) \cap E) \leq C \left(\frac{R}{r} \right)^\alpha \right\}$$

and the lower (Assouad) dimension is

$$\dim_L(E) = \sup \left\{ \alpha > 0 : \exists C, \rho > 0, \forall 0 < r < R \leq \rho, \inf_{x \in E} N_r(B(x, R) \cap E) \geq C \left(\frac{R}{r} \right)^\alpha \right\}.$$

Notice that these measure the “extreme” behaviour over $x \in E$, with $\dim_A E$ the “largest” and $\dim_L E$ the “smallest” growth exponents of the local covering numbers. Also notice that if E has an isolated point then $\dim_L E = 0$.

In a general metric space with metric d , $\dim_A(E) < \infty$ if and only if (E, d) is a *doubling* metric space.

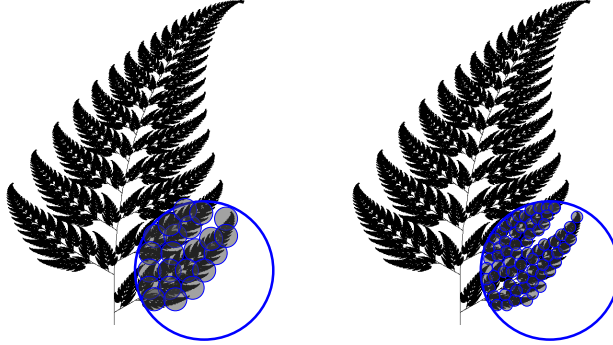


Fig. 4 Covering $E \cap B(x, R)$ by little balls $B(x_i, r)$ of two different sizes.

The variations on the Assouad dimension all impose further conditions on the relations between r and R . As an example, the *quasi-Assouad* dimension introduced in [19] is given by

$$\dim_{qA} E = \liminf_{\delta \rightarrow 0} \left\{ \alpha > 0 : \exists C, \rho > 0, \forall 0 < r < R^{1+\delta} \leq \rho, \sup_{x \in E} N_r(B(x, R) \cap E) \leq C \left(\frac{R}{r} \right)^\alpha \right\}.$$

There is also a dual lower version, $\dim_{qL} E$. The two-step process in defining $\dim_{qA} E$ makes it a bit unwieldy to work with.

It is possible to show that for any bounded $E \subset \mathbb{R}$

$$\dim_L(E) \leq \dim_{qL}(E) \leq \dim_H(E) \leq \underline{\dim}_B(E) \leq \overline{\dim}_B(E) \leq \dim_{qA}(E) \leq \dim_A(E).$$

Self-similar sets

It is worthwhile to mention that for many sets, including many self-similar sets, all the previous dimensions agree. Recall that a compact $F \subset \mathbb{R}$ is *self-similar* if there is a set of contractive similarities $w_i : \mathbb{R} \rightarrow \mathbb{R}$, $i = 1, 2, \dots, N$, so that

$$F = \bigcup_{i=1}^N w_i(F).$$

The collection $\{w_i\}_1^N$ is called an *iterated function system* (IFS) and we say that it has the *open set condition* if there is some non-empty open set $U \subset \mathbb{R}$ with $\bigcup_i w_i(U) \subseteq U$. Under this condition, $\dim_L F = \dim_A F$ and so all the dimensions coincide (see, for example, [10]).

5 Φ -dimensions

The most general of these variations are the Φ -dimensions $\dim_\Phi E$ which use a *dimension function* to specify the relation between r and R . In this setting, a dimension function is a map $\Phi : (0, 1) \rightarrow \mathbb{R}^+$ with the property that $t^{1+\Phi(t)}$ decreases to 0 as t decreases to 0. Given a dimension function Φ , the *upper Φ -dimension* is

$$\overline{\dim}_\Phi E = \inf \{ \alpha > 0 : \exists C, \rho > 0, \forall 0 < r < R^{1+\Phi(R)} \leq \rho, \sup_{x \in E} N_r(B(x, R) \cap E) \leq C \left(\frac{R}{r} \right)^\alpha \}. \quad (9)$$

As above, there is also a dual lower version, $\underline{\dim}_\Phi E$, where again $\underline{\dim}_\Phi E = 0$ anytime E has an isolated point, just like we mentioned above for the lower dimension $\dim_L E$. Notice the crucial use of Φ in this definition to control the size of r relative to R .

The Φ -dimensions were first introduced in [12], but then extensively investigated in [14]. We warn the reader that some authors (like those of [1]) use a slightly different definition and require instead $r = R^{1+\Phi(R)}$. This preserves many of the features of the Φ -dimensions but also causes some subtle differences. An example of

this is the monotonicity property from Proposition 6 part 2, which can be lost when $r \leq R^{1+\Phi(R)}$ is replaced by $r = R^{1+\Phi(R)}$.

Proposition 5 ([14]) *Let $E, F \subset \mathbb{R}$ be bounded.*

1. *If $E \subseteq F$ then $\overline{\dim}_\Phi E \leq \overline{\dim}_\Phi F$. However $\underline{\dim}_\Phi E \leq \underline{\dim}_\Phi F$ need not be true.*
2. *$\overline{\dim}_\Phi(E \cup F) = \max\{\overline{\dim}_\Phi E, \overline{\dim}_\Phi F\}$ and so $\overline{\dim}_\Phi$ is finitely stable.*
3. *$\min\{\underline{\dim}_\Phi E, \underline{\dim}_\Phi F\} \leq \underline{\dim}_\Phi(E \cup F) \leq \max\{\underline{\dim}_\Phi E, \underline{\dim}_\Phi F\}$.*

The next result gives more of the basic properties of the Φ -dimensions. Note that when Φ is “small enough” $\overline{\dim}_\Phi E$ coincides with the Assouad dimension, while if Φ is “big enough” then $\underline{\dim}_\Phi E$ coincides with the upper box dimension. Some of these relations are illustrated in Fig. 5. In interpreting the figure it is important to remember that “larger” Φ give smaller values for $\underline{\dim}_\Phi E$, see Proposition 6(2).

Proposition 6 ([14]) *Let Φ and Ψ be dimension functions.*

1. *$\dim_L E \leq \underline{\dim}_\Phi E \leq \underline{\dim}_B E \leq \overline{\dim}_B E \leq \overline{\dim}_\Phi E \leq \dim_A E$.*
2. *If $\Phi(t) \leq \Psi(t)$ for all small $t > 0$, then $\overline{\dim}_\Psi E \leq \overline{\dim}_\Phi E$ and $\underline{\dim}_\Phi E \leq \underline{\dim}_\Psi E$.*
3. *If $\Phi(t)/\Psi(t) \rightarrow 1$ as $t \rightarrow 0$, then $\overline{\dim}_\Phi E = \overline{\dim}_\Psi E$ and $\underline{\dim}_\Phi E = \underline{\dim}_\Psi E$ for all E .*
4. *If $\Phi(t) \rightarrow \delta > 0$ and E is a closed set, then $\underline{\dim}_\Phi E \leq (1 + \delta) \dim_H E$.*
5. *If $\Phi(t) \leq c/|\log(x)|$ then $\underline{\dim}_\Phi E = \underline{\dim}_L E$ and $\overline{\dim}_\Phi E = \dim_A E$.*
6. *If $\Phi(t) \rightarrow \infty$ then $\underline{\dim}_\Phi E = \underline{\dim}_B E$ and $\overline{\dim}_\Phi E = \overline{\dim}_B E$.*
7. *If $\Phi(t) \rightarrow 0$ as $t \rightarrow 0$, then*

$$\dim_L E \leq \underline{\dim}_\Phi E \leq \dim_{qL} E \leq \dim_{qA} E \leq \overline{\dim}_\Phi E \leq \dim_A E$$

and so the Φ -dimensions are between the quasi-Assouad and Assouad dimension.

There is another figure dual to Fig. 5 which would show similar relations for the lower Φ -dimensions. Notice the “phase transition cut-off” in the figure. The almost sure value of the upper Φ -dimension for i.i.d. random sets and measures exhibits a “phase transition” where there is one value for “large” dimension functions Φ and

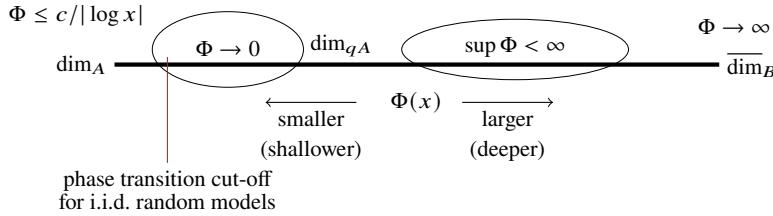


Fig. 5 Upper Φ dimension and the size of Φ .

Table 1 Summary comparison of dimensions (f is Lipschitz, g is bi-Lipschitz).

Property	$\dim_H$	$\dim_P$	$\underline{\dim}_B$	$\overline{\dim}_B$	$\underline{\dim}_\Phi$	$\overline{\dim}_\Phi$
$E \subset F \implies \dim E \leq \dim F$	✓	✓	✓	✓	X	✓
$\dim(E \cup F) = \max\{\dim E, \dim F\}$	✓	✓	X	✓	X	✓
$\dim(\cup_n E_n) = \sup_n \dim E_n$	✓	✓	X	X	X	X
$\dim f(E) \leq \dim E$	✓	✓	✓	✓	X	X
$\dim g(E) = \dim E$	✓	✓	✓	✓	✓	✓
$\dim(\text{closure}(E)) = \dim E$	X	X	✓	✓	✓	✓
$\dim O = n, \emptyset \neq O \subset \mathbb{R}^n$ open	✓	✓	✓	✓	X	✓

another for “small”, see [1, 15, 16, 17, 18, 20]. This transition happens according as to whether

$$\lim_{t \rightarrow 0} \frac{\Phi(t)}{\frac{\log |\log t|}{|\log t|}} \rightarrow 0 \text{ or } \infty.$$

It is unknown what happens if the limit is neither zero nor infinity. This phase transition is an indication that the Φ -dimensions between the Assouad and quasi-Assouad are particularly interesting.

It is also true that the quasi-Assouad dimension can be realized as a Φ -dimension, but the specific function Φ depends on the set E . That is, given a bounded $E \subset \mathbb{R}$, there is a dimension function Φ with $\Phi(x) \rightarrow 0$ as $x \rightarrow 0$, such that $\underline{\dim}_\Phi E = \underline{\dim}_{qA} E$. A similar statement also holds for the quasi-lower Assouad dimension.

Table 1 gives a summary comparison of the various dimensions that we have discussed. It does this in terms of some properties that the dimension does or does not always possess. Note that, quite surprisingly, a Lipschitz function can increase the Assouad dimension (see [11, Section 3.4]), and thus in general the upper Φ -dimension as well.

A very good way of understanding the difference between different dimension functions is via their associated *depth functions*. Suppose that (a_n) is a doubling sequence and let

$$s_n = \frac{1}{2^n} \sum_{i \geq 2^n} a_i$$

be the average length of the Cantor intervals in C_a at the n th level. Using this, we define the *depth function* $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ associate with Φ by the rule that

$$\varphi(n) = \min\{j \in \mathbb{N} : s_{n+j} \leq s_n^{1+\Phi(s_n)}\},$$

the idea being that for $R = s_n$, we need $r \leq s_{n+\varphi(n)}$. Thus $\varphi(n)$ counts the minimal number of “additional levels” you must descend in order to satisfy the condition $r \leq R^{1+\Phi(R)}$ at the scale $R = s_n$. Fig. 6 provides an illustration of this idea. Clearly if $\Phi(x) \leq \Psi(x)$ for all small $x > 0$ their associated depth functions will also satisfy $\varphi(n) \leq \psi(n)$. If $\varphi(n)$ is bounded (as is the case for the Assouad dimension) there is no requirement to look deep into the tree. On the other hand, if $\varphi(n)/n \rightarrow \infty$ then you look so deep into the tree that the resulting dimension is the box dimension.

For $s_n = 3^{-n}$ (like in the middle-1/3 Cantor set) and $\Phi(x) = \delta$ (as in the definition of $\dim_{qA} E$), we get $\varphi(n) = \delta n$. This means that as $R \rightarrow 0$, you must look deeper and deeper into the tree in order to compute $\dim_{qA} E$. The “cut-off” between small and large dimension functions turns out to be associated with $\varphi(n) \asymp \log(n)$.

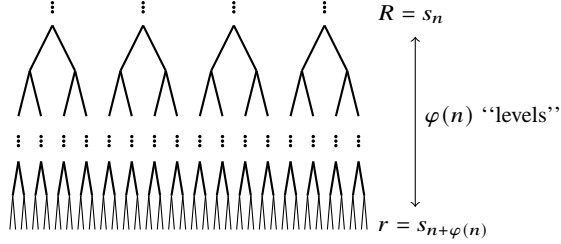


Fig. 6 The depth function $\varphi(n)$ associated with (a_n) and $\Phi(x)$.

In fact, for doubling (a_n) it is not too difficult to give an explicit formula for the Φ -dimensions in terms of the average tail-sums s_n and the depth function φ associated to Φ and C_a (see [14]):

$$\overline{\dim}_{\Phi} C_a = \inf \left\{ \beta : (\exists k_0, c_0 > 0) (\forall k \geq k_0, n \geq \varphi(k)) \left(\frac{s_k}{s_{k+n}} \right)^{\beta} \geq c_0 2^n \right\}$$

and

$$\underline{\dim}_{\Phi} C_a = \sup \left\{ \beta : (\exists k_0, c_0 > 0) (\forall k \geq k_0, n \geq \varphi(k)) \left(\frac{s_k}{s_{k+n}} \right)^{\beta} \leq c_0 2^n \right\}$$

Examples in [14] show that different dimension functions do measure different things, as described in the next two results. The key idea is as follows. Suppose that E is a central Cantor set with ratio of dissection r_k at step k such that there is an increasing sequence $n_k \in \mathbb{N}$ (maybe extremely sparse) with $r_k = \rho$ for $n_j + 1 \leq k \leq n_j + \varphi(n_j)$ and otherwise $r_k = \tau \leq \rho$, then $\overline{\dim}_{\Phi} E = \log 2 / |\log \rho|$.

Proposition 7 ([14]) *Suppose that Φ_1, Φ_2 are dimension functions decreasing to zero as $x \rightarrow 0$ and $|\log x| \Phi_2(x) \rightarrow \infty$ as $x \rightarrow 0$. Suppose further that there is some $\delta > 0$ with $\Phi_1(x) \geq (1 + \delta) \Phi_2(x)$ for small x . Then there are central Cantor sets E, F so that $\overline{\dim}_{\Phi_1} E < \overline{\dim}_{\Phi_2} E$ and $\underline{\dim}_{\Phi_1} F > \underline{\dim}_{\Phi_2} F$.*

With very careful control of the scaling ratios in the construction of a central Cantor set, we can even exactly specify $\dim_{\Phi} C$ for a continuous family of dimension functions Φ . This is a very strong indication that different dimension functions measure different geometric properties of a set.

Proposition 8 ([14]) *Assume that for each $p \in (0, 1)$, Φ_p is a dimension function decreasing to zero as $x \rightarrow 0$ and $|\log x| \Phi_p(x) \rightarrow \infty$ as $x \rightarrow 0$. Assume also that*

$$\Phi_p(x)/\Phi_q(x) \rightarrow \infty \text{ as } x \rightarrow \infty \text{ whenever } p > q.$$

Choose any $0 < \alpha < \beta < 1$ and suppose $d : (0, 1) \rightarrow [a, b]$ is monotonically decreasing and continuous. Then there is a central Cantor set C with

$$\overline{\dim}_{\Phi_p} E = d(p) \quad \text{for each } p \in (0, 1).$$

A similar statement holds for the lower Φ dimension.

Closing thoughts

All of the dimensions that we have discussed are distinct and thus together provide a more refined description of a set than would be possible with only one of them. The Φ -dimensions in particular give great flexibility because of the freedom in choosing Φ . So far the major application of this flexibility has been in understanding the threshold phenomena in iid random models, but we look forward to seeing other applications in quite different domains.

There are also versions of all of the preceding dimensions, including the Φ -dimensions, for measures and these dimensions can similarly be used to quantify the “smoothness” of the measure.

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