A Survey of Multidimensional Radix Representations

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Outline

Radix Representations

Examples:

- Decimal: Base/radix $A = 10$, digit set $D = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- Binary: Base/radix $A = 2$, digit set $D = \{0, 1\}$

Digit set can be any set of coset representatives of $\mathbb{Z}/A\mathbb{Z}$:

- $A = 3$, $D = \{-1, 0, 1\}$ or $D = \{-3, 1, 8\}$
- $A = 10$, $D = \{-4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$

Radix can be any integer with $|A| \geq 2$:

- $A = -2$, $D = \{0, 1\}$

Multidimensional Radix Representations

- A a *dilation matrix*: $A \in M_m(\mathbb{Z})$, $|\lambda| > 1$ for all eigenvalues
- D a *digit set*: $D \simeq \mathbb{Z}^m / A\mathbb{Z}^m$, $D \subset \mathbb{Z}^m$ a complete set of coset representatives
  - Note: $|D| = |\det A|$

Definition

A vector $x \in \mathbb{R}^m$ has a radix representation with base A and digit set D if there is a finite $n = n(x)$ and a sequence of digits $d_j \in D$ such that

$$x = \sum_{j=0}^{n} A^j d_j + \sum_{j=-\infty}^{-1} A^j d_j.$$

We say that (A, D) yields a radix representation for $\mathbb{Z}^m$ if every $x \in \mathbb{Z}^m$ has a radix representation (with only positive powers of A).

Example - Multidimensional Radix Representation

- $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, the *twin dragon* matrix
- $D = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

Example in Sage

Number Systems for Lattices

- Λ a lattice
- A an endomorphism of Λ with nontrivial cokernel
- D a complete set of coset representatives of $\Lambda/A\Lambda$

Definition

A triple (Λ, A, D) is called a number system if for every $x \in \Lambda$ there exists a finite $n = n(x)$ and sequence of digits $d_j \in D$ such that

$$x = \sum_{j=0}^{n} A^j d_j.$$

- Not all of our previous examples are number systems.

Dynamical Systems

Definition

A dynamical system is a quadruple $(X, \mathcal{F}, \rho, T)$, where X is a non-empty set, $\mathcal{F}$ is a σ-algebra on X, ρ is a probability measure on $(X, \mathcal{F})$, and $T : X \to X$ is a surjective ρ-measure preserving transformation.

For $x \in X$, the sequence $x, Tx, T^2x, \ldots$ is called the T-orbit of x.

- Example: $X = [0, 1]$; $\mathcal{F} = \mathcal{B}$, the Borel σ-algebra; $\rho = \lambda$, Lebesgue measure on $[0, 1]$; and $Tx = \frac{x}{2}$ or $\frac{x+1}{2}$ with probabilities $\left(\frac{1}{2}, \frac{1}{2}\right)$.
- Some orbits for this dynamical system in Sage

Digit Process on $T(A,D)$

Definition

Let A be a dilation matrix and D a digit set for A.

$$T(A,D) = \left\{ x \in \mathbb{R}^m : \; x = \sum_{j=1}^{\infty} A^{-j} d_j, d_j \in D \right\}.$$

Define a transformation $S : T(A,D) \to T(A,D)$ by

$$Sx = A^{-1}(x + \xi),$$

where ξ is chosen from D according to some probability distribution.

- Example: ξ chosen uniformly from D. Then S is an ergodic transformation.
- $(T(A,D), \mathcal{B}, \lambda, S)$ is a dynamical system.

Digit Process on Sequence Space

Let

$$\Omega = \prod_{j=1}^{\infty} D = \{(x_1, x_2, x_3, \dots) : x_j \in D\}.$$

Define $S' : \Omega \to \Omega$ by

$$S'(x_1, x_2, x_3, \dots) = (\xi, x_1, x_2, x_3, \dots),$$

with ξ chosen from D by the same probability distribution.

- When ξ is chosen uniformly from D, S' is the canonical right shift operator on Ω.
- $(\Omega, \mathcal{C}, \rho, S')$ is a dynamical system, with $\mathcal{C}$ the σ-algebra of cylinder sets on Ω and ρ the measure that gives a cylinder of length n weight $|D|^{-n}$.

Equivalence

- $(\Omega, \mathcal{C}, \rho, S')$ and $(T(A, D), \mathcal{B}, \lambda, S)$ are isomorphic dynamical systems.
- Whenever a dynamical system is equivalent to a system like $(\Omega, \mathcal{C}, \rho, S')$, we can think of S' as being a shift on sequences of digits. Other examples include continued fractions, β-transformations, GLS transformations.
- This isomorphism allows us to relate multidimensional radix representations to one-dimensional radix representations with the same size digit set.

Hot Spots and Normal Numbers - Definitions

Definition

A vector $x \in T(A,D)$ is called a base-A hot spot if there exists a constant vector $\alpha \in T(A,D)$ such that

$$\liminf_{h\to 0}\liminf_{n\to\infty} \frac{\#_{0\le j<n}[\{A^j\alpha\} \in B_h(x)]}{n\lambda(B_h)} = \infty$$

where $B_h(x)$ is the ball of radius h centered at x in $\mathbb{R}^m$, and λ is Lebesgue measure in $\mathbb{R}^m$.

Definition

A vector $x \in T(A,D)$ is A-normal if the digits in the radix representations of x are uniformly distributed in D.

Hot Spots, Normality, and Randomness

Theorem (Hot Spot Theorem)

(Bailey-Rudolph, generalized to higher dimensions by Curry) A vector $x \in T(A, D)$ is A-normal if and only if it has no base-A hot spots.

- There are some connections here to random numbers, for certain definitions of randomness.

When is (Λ, A, D) a Number System?

Necessary conditions:

Proposition (A. Kovács)

If (Λ, A, D) is a number system, then

1. *D must be a complete set of residues modulo A,*
2. *A must be expansive, and*
3. $\det(I - A) \neq \pm 1$

Sufficient Conditions: Choice of Digit Set

Theorem (Curry)

Let A be a dilation matrix for $\mathbb{Z}^m$. If

$$\min\{\sigma : \ \sigma \text{ a singular value of } A\} > 2$$

then $(\mathbb{Z}^m, A, D)$ is a number system with the centered digit set $D = AF \cap \mathbb{Z}^m$, where $F = \left(-\frac{1}{2}, \frac{1}{2}\right]^m$.

Theorem (Germán and Kovács)

For a given endomorphism A, let ρ denote the spectral radius. If $\rho(A^{-1}) < \frac{1}{2}$ then there exists a digit set D for which (Λ, A, D) is a number system.

In particular, D is formed by choosing a representative of minimal norm for each coset of $\Lambda / A\Lambda$.

Exact Conditions

Theorem (Curry)

$(\mathbb{Z}^m, A, D)$ is a number system if and only if:

(i) *$T(A, D)$ tiles $\mathbb{R}^m$ under translation by $\mathbb{Z}^m$, and*

(ii) *0 is in the interior of $T(A, D)$.*

Decision Algorithm:

- The Euclidean algorithm is a contraction until orbits enter a ball of a certain finite radius (upper bounds on the radius can be easily computed).
- Once inside the ball, there are only finitely many points. Orbits must either arrive at 0, or must fall into non-zero cycles.
- $(\mathbb{Z}^m, A, D)$ is a number system if and only if all orbits arrive at 0 (no non-zero cycles).

Relationship between A and D

Question

For a given lattice Λ and endomorphism A, how many digits sets D are there for which (Λ, A, D) is a number system?

Question

Can the digits sets for which (Λ, A, D) is a number system be characterized?

- Some results in one dimension (Odlyzko, Matula, Lagarias-Wang).

What is the Length of Expansions?

Proposition (A. Kovács)

There is a constant c for which

$$\ell(x) \leq \frac{\log \|x\|}{\log\left(1/\|A^{-1}\|\right)} + c.$$

Pseudodigits

If (Λ, A, D) is *not* a number system, then some orbits fall into non-zero periodic sequences.

Definition

A pseudodigit is a representative of a non-zero periodic orbit in $(\mathbb{Z}^m, A, D)$.

Question

Can we characterize the number and properties of these non-zero periodic orbits?

Iterated Function Systems

A a dilation matrix for $\mathbb{Z}^m$.

For each digit $d \in D$, define a function on $\mathbb{R}^m$,

$$f_d(x) = A^{-1}(x + d).$$

The collection $\{f_d\}_{d \in D}$ defines an *iterated function system* (IFS).

We can use the individual functions f_d to define a set mapping

$$F(K) = \bigcup_{d \in D} f_d(K)$$

on subsets $K \subset \mathbb{R}^m$. The sequence $\{F^j(K)\}$ will converge in Hausdorff measure to a fixed point, called the *attractor* of the IFS (eg. suppose K compact).

Attractors

- The attractor is compact
- The attractor is a fractal
- The attractor is $T(A, D)$

$$\text{Recall: } T(A, D) = \left\{ x = \sum_{j=1}^{\infty} A^{-j} d_j, d_j \in D \right\}$$

Basic Properties of $T(A, D)$

- Self-affine:
  $T(A, D) = F(T(A, D)) = \cup_{d \in D} A^{-1}(d + T(A, D))$
- If $0 \in D$, then $T(A, D)$ satisfies the *open set condition* (essentially: has non-empty interior) – **assume $\mathbf{0} \in \mathbf{D}$**
- Tiling: $T(A, D)$ tiles $\mathbb{R}^m$ under translation by a sub-lattice of $\mathbb{Z}^m$
  - The Lebesgue measure is an integer
  - When $\lambda(T(A, D)) = 1$, $T(A, D)$ tiles under translation by $\mathbb{Z}^m$, $T(A, D)$ is a fundamental domain for the integer lattice

Tiling and Radix Representations

Theorem (Curry)

Let A be a dilation matrix for $\mathbb{Z}^m$ and let D be a complete digit set for A. If $(\mathbb{Z}^m, A, D)$ is a number system, then $T(A, D)$ tiles $\mathbb{R}^m$ under translation by the full lattice $\mathbb{Z}^m$.

Note: A and D may not yield a radix representation for all of $\mathbb{Z}^m$ yet $T(A, D)$ may still have measure 1 and tile under translation by $\mathbb{Z}^m$.

- Example: $A = 2, D = \{0, 1\}, T(A, D) = [0, 1]$
- Example: the twin dragon

Connectedness

Question

Given a dilation matrix A for $\mathbb{Z}^m$, does there exist a digit set D for which $T(A,D)$ is connected?

- *Can such digit sets be characterized?*
- *How does structure of A affect chances for connectedness?*

Proposition

If A is "not too skew", then $T(A,D)$ is connected for D the centered digit set,

$$D = AF \cap \mathbb{Z}^m, \quad F = \left(-\frac{1}{2}, \frac{1}{2}\right]^m.$$

Neighbors and Connectedness

- Iterative approach to connectedness:
  - Start with a simple set (eg. $[-1/2, 1/2]^m$) and look at iterates under the set mapping F
  - Neighbor graph (Hata, Scheicher-Thuswaldner, Gröchenig-Haas, others)

- Boundary approach:
  - $T(A, D)$ connected only if its boundary is connected
  - (Lau, Ngai, others)

Some Sufficient Conditions for Connectedness

Theorem (Curry-Laarakker)

If A and B are similar matrices, $B = PAP^{-1}$, with $P \in M_m(\mathbb{Z})$, then $D_B = PD \subset \mathbb{Z}^m$ is a digit set for B, and $T(A, D)$ is connected if and only if $T(B, D_B)$ is connected.

In particular, if the dilation matrix A is similar to its Jordan form J over the integers, $A = QJQ^{-1}$, then we may set $D = QD_J$ with D_J the centered digit set for J to find a digit set for A such that $T(A, D)$ is connected.

Theorem (Curry-Laarakker)

Let S_{AF} be the set of edge-neighbors of the parallelepiped AF in $\mathbb{R}^m$. Consider the lattice $\mathbb{Z}^m$ as a graph, where two points x and y have an edge between them if they are distance 1 apart. If, for each $g \in S_{AF}$, the set $(AF \cap (g + AF)) \cap \mathbb{Z}^m$ is a connected subset of $\mathbb{Z}^m$ considered as a graph, then $T(A, D)$ is connected.

Disc-Like $T(A, D)$

Question

For which A and D is $T(A, D)$ disc-like (homeomorphic to a ball in $\mathbb{R}^m$)?

- Most investigation has considered the boundary of $T(A, D)$
- (Lau, Bandt-Wang, Luo-Zhou, others)

Wavelet Connection

Sets $T(A, D)$ were first considered as support for scaling functions for multivariable wavelets (Gröchenig-Haas).

Definition

A wavelet is a function $\psi(x)$ whose dilates and translates form an orthonormal basis for L^2.

Specifically: let A be a dilation matrix for $\mathbb{Z}^m$. $\psi(x)$ is a wavelet if the set $\{|\det A|^{j/2}\psi(A^j x - k), j \in \mathbb{Z}, k \in \mathbb{Z}^m\}$ forms an orthonormal basis for $L^2(\mathbb{R}^m)$.

Example: Haar wavelet.

Multiresolution Analysis

Definition

A multiresolution analysis (MRA) is a nested sequence of subspaces of $L^2(\mathbb{R}^m)$,

$$\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots ,$$

satisfying the following conditions.

1. $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^m)$,
2. $\cap_{j\in\mathbb{Z}} V_j = \{0\}$,
3. $f(x) \in V_0$ *if and only if* $f(x-k) \in V_0$ *for all* $k \in \mathbb{Z}^m$,
4. $f(x) \in V_j$ *if and only if* $f(Ax) \in V_{j+1}$ *for all* $j \in \mathbb{Z}$, *and*
5. *there exists a function ϕ, called a scaling function, such that the set*

   $$\{\phi(x-k), k \in \mathbb{Z}^m\}$$

   is an orthonormal basis for V_0.

Scaling Equation and Haar Scaling Functions

The scaling equation:

$$\phi(A^{-1}x) = \sum_{k \in \mathbb{Z}^m} c_k \phi(x - k)$$

for some *scaling coefficients* c_k.

Theorem (Gröchenig-Madych)

If $T(A, D)$ has m-dimensional Lebesgue measure 1 (equivalently, if $T(A, D)$ tiles $\mathbb{R}^m$ under translation by $\mathbb{Z}^m$), then the indicator function of this set, $\mathbf{1}_{T(A,D)}(x)$ is a scaling function for a multiresolution analysis.

Low-Pass Filters

Taking Fourier transforms, the scaling equation becomes:

$$\hat{\phi}(\xi) = m(A^{*-1}\xi)\hat{\phi}(A^{*-1}\xi),$$

where $m(\xi)$ is a function periodic on $T(A, D)$ (or any fundamental domain of $\mathbb{Z}^m$).

Definition

The function $m(\xi)$ is called the low-pass filter for the scaling function ϕ and associated MRA.

Question

How to characterize low-pass filters?

A Characterization Theorem

Theorem (Gundy, generalized by Curry)

(Part 1) If $m(\xi)$ is a low-pass filter associated with an MRA, then the following hold.

1. $m(\xi)$ *is* $\mathbb{Z}^m$*-periodic, and* $\lim_{j\to\infty}|m(A^{*-j}\xi)|^2 = 1$.
2. *The operator* $\mathbf{p} : f(\xi) \mapsto |m(A^{*-1}\xi)|^2 f(A^{*-1}\xi)$ *has a non-trivial fixed point* $|\hat{\phi}(\xi)|^2 \in L^1 \cap L^2(\mathbb{R}^m)$*, and the operator* $\mathcal{P} : f(\xi) \mapsto \sum_{d\in D}|m(A^{*-1}(\xi + d))|^2 f(A^{*-1}(\xi + d))$ *has a non-trivial fixed point* $e(\xi) = \sum_{k\in\mathbb{Z}^m}|\hat{\phi}(\xi + k)|^2 \in L^\infty(\mathbb{T}^m$.
3. $e(\xi)$ *is the unique fixed point of* $\mathcal{P}$ *in a certain class of functions,* $D_\infty(\hat{\phi})$.

Characterization Theorem Cont.

Theorem (Part 2)

Conversely, if a function $m(\xi)$ satisfies these conditions, then we can construct a low-pass filter for a multiresolution analysis from $m(\xi)$.

The Important Details

From the scaling equation on the Fourier transform side, we can write $\hat{\phi}$ as an infinite product, $\hat{\phi}(\xi) = \prod_{j=1}^{\infty} m(A^{*-j}\xi)$. For convergence, we need $\lim_{j\to\infty}|m(A^{*-1}\xi)| = 1$.

From the Plancherel formula, we can derive that $\sum_{d\in D}|m(A^{*-1}(\xi + d))|^2 = 1$ almost surely. This lets us use $p(\xi) = |m(\xi)|^2$ to define a transition function on a Markov process:

- Randomly choose some $x \in T(A, D)$ and set $X_0 = x$.
- Let $X_{j+1} = A^{*-1}(X_j + d_j)$, where $d_j = d \in D$ with probability $p(A^{*-1}(X_j + d))$.

Boundary Behavior

The characterization theorem implies that $\lim_{j\to\infty} X_j = 0$ almost surely. Thus, transition functions $p(\xi)$ that generate processes with the same limiting behavior should allow us to construct low-pass filters.

- Given a transient,irreducible Markov process on a set S (some restrictions apply, some conditions can be relaxed), a *Martin boundary* can be appended to S to complete the space under a certain metric.
- Harmonic functions $u(x)$ on S can then be represented in the form $u(x) = \int_{\partial_m S_M} K(x,\alpha)\mu(d\alpha)$ for a Martin kernel K (related to a Green's function) and measure μ.

  Conceptually: this is a generalized mean value property for harmonic functions in terms of values on the (extended) boundary of the set S.

Martin Boundary of Random Walks on Fractals

- Martin boundary theory has experienced a small resurgence in the past couple years, as a tool for studying the limiting behavior of random walks on fractals, such as the Sierpinski gasket.

Martin Boundary and Low-Pass Filters

Question

The Martin boundary of a Markov process sets up a duality between the exit measure of the process and sets in a boundary on which the measure is supported. Can this framework allow us to understand low-pass filters better?

Question

Can the Martin or similar boundaries for the Markov process give useful information about the topology of $T(A, D)$?

Thanks for your attention!

For further details on any of the topics covered, please ask me for references.