Proof in Mathematics Education

Research, Learning and Teaching

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CHAPTER 10

ARGUMENTATION STRUCTURES

In this chapter we describe argumentation structures, which are the global, overall structures of the arguments that occur in a classroom proving process. These structures, first described by Knipping (2003b) allow one to examine the entirety of the proving process, without losing sight of the details that make it up. Understanding the rationales and the contextual constraints that shape theseargumentations can help us to improve our efforts in teaching proof, by giving us greater insight into the nature of existing proving processes in classrooms.

As you read this chapter you may want to reflect on these questions:

- What teaching goals seem to be indicated by the different structures identified?
- What constraints in schools or in teaching generally might influence the shapes of these structures?
- How much variation would you expect to find between the structures identified in different teachers' classrooms compared to the variation you would find between different lessons taught by the same teacher or between different national school systems and cultures of teaching?

TOULMIN'S FUNCTIONAL MODEL AND ARGUMENTATION STRUCTURES

Analyses of argumentations in mathematics lessons have received increasing attention in recent years (Knipping, 2003b; Krummheuer, 1995, 2007; Pedemonte, 2007) and have become a means to better understand proving processes in class. As discussed in Chapter 8 the work of Toulmin has provided researchers in mathematics education with a useful tool for research, including arguments in classrooms (Knipping, 2003b; Krummheuer, 1995) and individual students' proving processes (Pedemonte, 2002).

Toulmin (1958) describes the basic structure of rational arguments as a linked pair datum + conclusion (see Figure 17). This step might be challenged and so it is often explicitly justified. A 'warrant' is given to establish the "bearing on the conclusion of the data already produced" (p. 98). These warrants "act as bridges, and authorize the sort of step to which our particular argument commits us" (p. 98). While Toulmin acknowledges that the distinction between data and warrants may not always be clear, their functions are distinct, "in one situation to convey a piece of information, in another to authorise a step in an argument" (p. 99). In fact, the same statement might serve as either datum or warrant or both at once, depending on context (p. 99), but according to Toulmin the distinction between datum, warrant, and the conclusion or claim provides the elements for the "skeleton of a pattern for analyzing arguments" (p. 99).

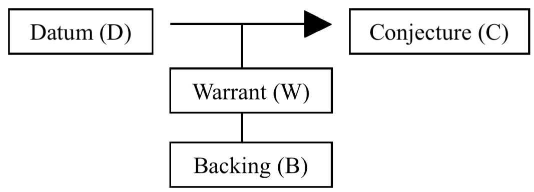


Figure 17. Toulmin model.

Toulmin adds several other elements to this skeleton, only one of which will be discussed here. Both the datum and the warrant of an argument can be questioned. If a datum requires support, a new argument in which it is the conclusion can be developed. If a warrant is in doubt, a statement Toulmin calls a "backing" can be offered to support it.

Toulmin's functional model of argumentation has been a foundation of Knipping's work (2001, 2002, 2003ab, 2004, 2008). But she also pointed out the need for a model that would also allow the structure of the argument as a whole to be laid out. As Toulmin notes "an argument is like an organism. It has both a gross, anatomical structure and a finer, as-it-were physiological one" (Toulmin, 1958, p. 94). Whereas Toulmin's aim was to explore the fine structure, Knipping attempted to extend the Toulmin model and to provide a model that also allows one to describe the gross structure, which she calls the *global argument*. The method she proposed for reconstructing arguments in classrooms is presented in Knipping (2008). Very briefly, the analyses of proving discourses use the Toulmin model in order to identify individual steps from data to conclusion. As the conclusions of some steps are recycled as data for others, these steps join up into *argumentation streams* (AS); however, these streams are generally not linear chains of steps. Argumentation streams themselves are interconnected in more complex ways and together form the *argumentation structure*. The analysis proceeds from the fine structure captured in individual steps to the global structure of the entire argumentation.

In the following four different types of argumentation structures that occurred in classroom proving processes will be presented and discussed. The first two types Knipping (2003ab) called the *source-structure* and the *reservoir-structure*. They were observed in proving processes from German and French classrooms. The last two are from a classroom in Canada. We refer to them as the *spiral-structure* and the *gathering-structure*.

THE SOURCE-STRUCTURE

In proving discourses with a source like argumentation structure, arguments and ideas arise from a variety of origins, like water welling up from many springs. The structure has these characteristic features:

– Argumentation streams that do not connect to the main structure.
– Parallel arguments for the same conclusion.

- Argumentation steps that have more than one datum, each of which is the conclusion of an argumentation stream.
- The presence of refutations in the argumentation structure.

The source-structure is also characterised by argumentation steps that lack explicit warrants or data. While this also occurs in the other types of argumentation structure we will examine, it is frequent in the source-structure.

The teacher encourages the students to formulate conjectures which are examined together in class. In some cases this means that students propose conjectures which are unconnected to the overall structure. More than one justification of a statement is appreciated and encouraged by the teacher. This diversity of justifications results in an argumentation structure with parallel streams in which intermediate statements are justified in various ways. False conjectures are eventually refuted, but they are valued as fruitful in the meantime.

In argumentations with a source-structure a funnelling effect becomes apparent. Towards the end of the argumentation only one chain of statements is developed in contrast to the beginning where many parallel arguments are considered. For example, in Figure 18 only a single chain of arguments (AS-7) occurs in the second half of the argumentation. The argumentation begins in a very open way, drawing on many sources, but is funnelled towards one final conclusion. Thus a variety of justifications all support the overall argument.

Example 1: The source-Structure in Mr. Lüders' Class

Our first example of the source-structure comes from a grade 9 mathematics class in Germany where the teacher, Mr. Lüders, sought to develop a proof of the Pythagorean Theorem together with the class (Knipping, 2003b). The argumentation structure (Figure 18) includes the features typical of the source-structure.

- Argumentation streams that do not connect to the main structure (AS-6)
- Parallel arguments for the same conclusion (AS-3, AS-4 and AS-5).
- Argumentation steps that have more than one datum, each of which is the conclusion of an argumentation stream (in AS-5 and AS-7).
- The presence of refutations in the argumentation structure (in AS-2 and AS-6).

Argumentation stream AS-6 is an example of two of these features: refutations, and argumentation streams that do not connect to the overall structure. In it Sebastian conjectures, without providing any supporting data or warrant, that the area of the rectangles in the right hand figure in the proof diagram (Figure 19) is equal to the area of the inner squares. This argument was refuted by the teacher and was never integrated into the larger structure.

The parallel arguments are AS-3, AS-4, and AS-5. They all lead to the conclusion that the angle γ of the inner quadrilateral (in the left hand diagram in Figure 19) is 90° on the basis of both visual and conceptual arguments. In AS-3 there is data missing and one step of AS-5 lacks a warrant.

We will have a closer look at argumentation stream AS-2 that precedes these parallel arguments and which includes another refutation. This will provide an illustration of the process by which the classroom discourse is represented in an

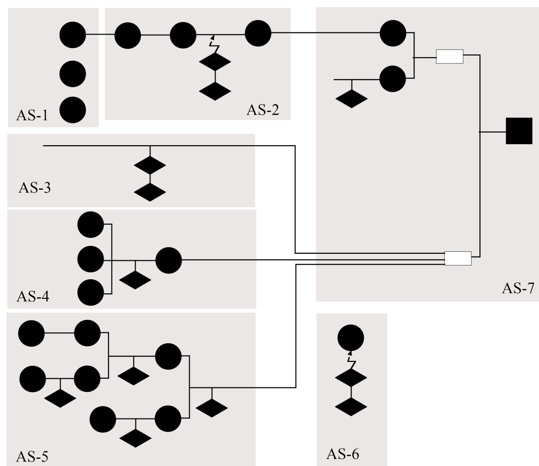


Figure 18. Overall argumentation structure in the proving process in Mr. Lüders' class.

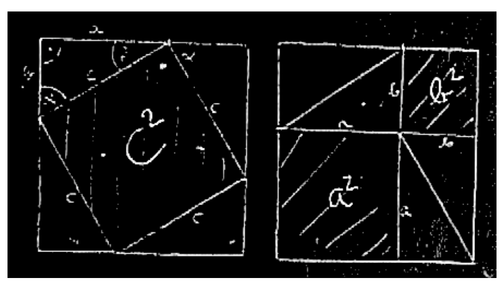


Figure 19. Proof diagram from Lüders' class.

argumentation structure diagram. For more details on this process see Knipping (2008). AS-2 begins when the teacher asks why the inner quadrilateral in the left hand figure is a square and Stefanie gives a reason, but an insufficient one.

35 Teacher: A square in a square. This is a square in a square, can you tell me, why this is a square? … Why is this a square, why is this a square? You can tell me anything. Stefanie!

38 Stefanie: Because it has four sides of equal length?

39 Teacher: Actually I believe you. So, if this is side b and this is side a and … And how long is this side?

41 Student: c too.

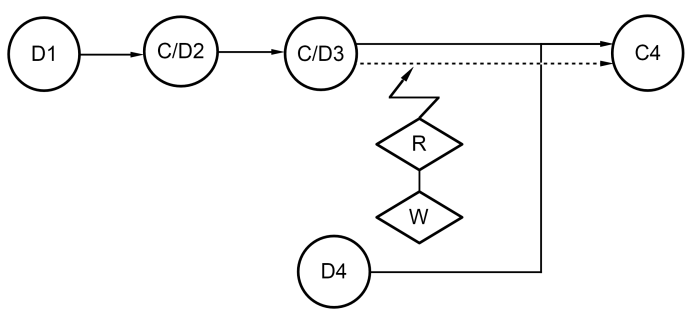
42 Teacher: c too, right? I have always taken the same triangle. Your reasons are fine. So, what else? Katrin, pay attention please. ... Is Stefanie's justification sufficient for proving That this is a square?

45 Student: No.

46	Torben:	It also has four right angles?
47	Teacher:	Why does it have four right angles? … You claim that this is a right angle?
52	Teacher:	... Again, why is this a right angle? Four equal sides, everybody who has ever eaten salinos [a rhombus shaped liquorice] knows this, are not enough to form a square. ... Eric, you start.

(Knipping, 2003b, p.155, ellipses in original represent pauses)

Figure 20 shows the structure of this argumentation stream. Note that the elements of it do not occur in chronological order. The teacher's argument that the sides of the inner quadrilateral are equal to the hypotenuse *c* of the triangle with sides *a* and *b* (39–42) comes after Stefanie's argument that the quadrilateral in the square is a square, because it has four sides of equal length (38) but in the structure of the argument it precedes it because it provides data she needs for her argument. In this way the statement "it has four sides of equal length" (38), which is assumed by Stefanie is supported. Stefanie's conclusion, however, is questioned: "IsStefanie's justification sufficient for proving that this is a square?" (43) and finally refuted by the teacher who provides a reason from an everyday context. "Four equal sides, everybody who has ever eaten salinos knows this, are not enough to form a square." (52/53). Here a mathematical refutation is supported by a backing from an everyday context. Torben's question suggests the piece of data that is missing from Stefanie's argument: "It also has four right angles?" (46). The teacher, by asking for a justification why the angle is a right angle, implicitly turns Torben's question into data.



D1: a and b, segments of the outer square, are sides of congruent triangles
C/D2: The sides of the inner figure are the sides c of the congruent triangles
C/D3: The inner figure has four equal sides of length c
R: Four equal sides is not a sufficient condition for a figure to be a square
W: Salinos are an example of a figure that have four equal sides but are not square
C4: The inner figure is a square
D4: The inner figure has four right angles

Figure 20. Argumentation stream AS-4 from Lüders' class.

Example 2: The Source-Structure in Nissen's Class

The following example of the source-structure was observed in another German grade 9 mathematics class, taught by Ms. Nissen (Knipping, 2003ab). To begin the proving process she sketched a drawing (Figure 21) on the chalkboard and asked her students to interpret it.

Eight different argumentation streams (AS-1 to AS-8) make up the global argumentation structure (Figure 22). Again the typical features of the source-structure are evident:

– Argumentation streams that do not connect to the main structure (AS-6)
– Parallel arguments for the same conclusion (AS-1 and AS-2).
– Argumentation steps that have more than one datum, each of which is the conclusion of an argumentation stream (AS-8).
– The presence of refutations in the argumentation structure (AS-3, AS-6).

AS-1 and AS-2 are parallel argumentation streams for the conclusion that the side of the outer square is *c*. In AS-1 the argument is based on the conclusion that the inner quadrilateral is a square, with the drawing of the proof figure as a warrant. In AS-2 it is argued that the triangles make up the outer shape, again based on the drawing.

Visually, argumentation steps which have more than one datum, each of which is the conclusion of an argumentation stream, show up as long verticals connecting several data with a conclusion. For example the conclusions of AS-4, AS-5 and AS-7 are data for the first step in AS-8.

There are two interesting refutations within this argumentation structure. In AS-3 Maren, in the process of describing the area of the outer square c^2, assumes that the area of the inner square is b^2. The teacher contradicts her, refuting Maren's suggestion visually. She then develops together with the class an argument that the side length of the inner square is $b–a$ (AS-5) and therefore the inner square's area must be $(b–a)^2$. This provides the data for a justification that c^2 consists of a square with side $b–a$ and four congruent right triangles, which is noted on the blackboard as follows: $c^2=(b\text{-}a)^2+4rwD$.

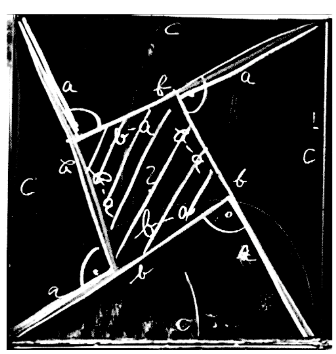


Figure 21. Proof diagram from Nissen's class.

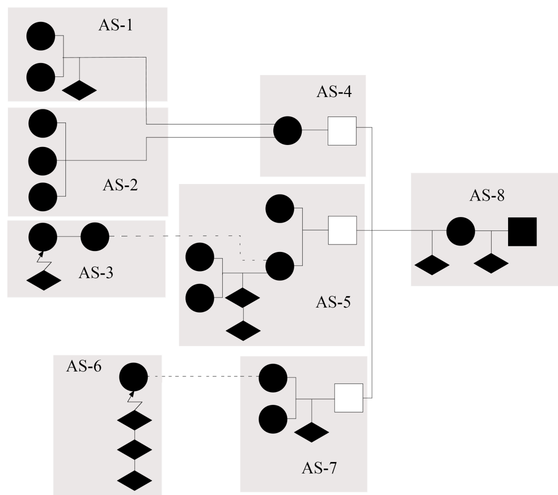


Figure 22. The source-structure in Nissen's class.

Sascha's conjecture (AS-6) is the other example of a refutation. Sascha claims that two of the triangles form a square which the teacher refutes by having the class put together the cut-out triangles. However, she says "I really like your idea, I think ideas that lead to the right result in detours are wonderful" giving value to Sascha's conjecture even though she has refuted it. This refutation provides a context for the argument in AS-7 that two of the triangles form a rectangle.

THE RESERVOIR-STRUCTURE

Argumentations with a reservoir-structure flow towards intermediate target-conclusions that structure the whole argumentation into parts that are distinct and self-contained. The statements that mark the transition from the first to the second part of the proving discourse (shown as rectangles) are like reservoirs that hold and purify water before allowing it to flow on to the next stage. Most of the features listed above as characteristic of the source-structure are missing in the reservoir-structure, with the exception of argumentation steps which have more than one datum each of which is the conclusion of an argumentation stream. Argumentation steps that lack explicit warrants or data occur, but less often than in the source-structure.

The most important feature of the reservoir-structure, which distinguishes it from a simple chain of deductive arguments, is that the reasoning sometimes moves backwards in the logical structure and then forward again. Initial deductions lead to desired conclusions that then demand further support by data. This need is made explicit by identifying possible data that, if they could be established, would lead to the desired conclusion (indicated by the dotted line in Figure 23). Once these data are confirmed further deductions lead reliably to the desired conclusion.

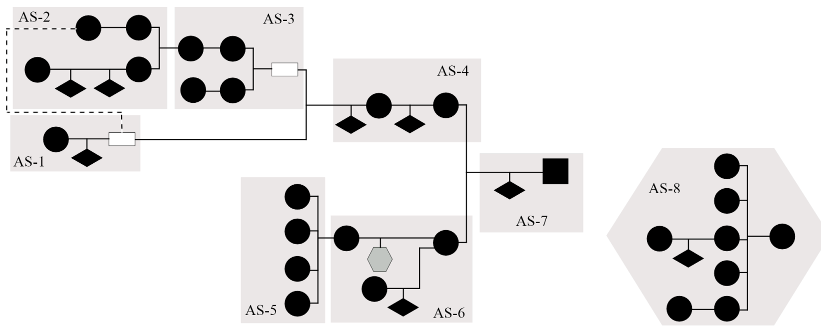


Figure 23. The reservoir-structure in Pascal's class.

This characterises a self-contained argumentation-reservoir that flows both forward towards, and backwards from, a target-conclusion.

Example 3: The Reservoir-Structure in Pascal's class

This example comes from a French level 4 (age 13–14) classroom where the proof of the Pythagorean Theorem is the topic. We have seen it before, in Chapter 6, where we used it as an example of abductive reasoning. We will now consider how it fits into the larger argumentation structure.

The class has concluded (in AS-1) that the inner quadrilateral of the proof diagram (see Figure 24) is a rhombus. They make an abduction from the desired result that ABCD is a square, the datum that ABCD is a rhombus, and the general rule that if a rhombus has a right angle it is a square, to conclude that ABCD has a right angle. This becomes the target-conclusion in the argumentation streams AS-2 and AS-3. The three streams AS-1, AS-2 and AS-3 form a reservoir in which the argumentation remains until it is sufficiently clarified to proceed.

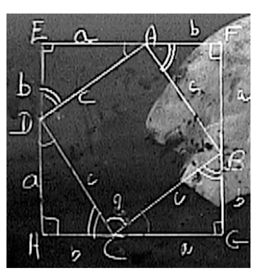


Figure 24. Diagram used in Pascal's class for Pythagorean Theorem proof.

A closed structure can also be found in the second part of the process, formed by AS-5, AS-6 and AS-7. In contrast to the reservoir in the first part, the argumentation in the second part only flows forwards. In AS-5 it is justified that the area of the outer square equals $(a+b)^2–2ab$. The subsequent argument (AS-6) restricts and directs the argumentation towards the final target-conclusion, that $a^2+b^2=c^2$ (AS-7).

AS-8 represents a variation on this theme. At the end of the overall argumenttation a student asks for a justification of the statement $(a+b)^2=a^2+2ba+b^2$ which is used as a warrant in the argumentation. The teacher responds to a student's question by reminding the class that the statement has been proven in the previous lesson, but proves the statement again (AS-8), together with the students. In this case a warrant is needed in AS-6, and this motivates the deduction of a suitable warrant in AS-8.

Example 4: The Reservoir-Structure in Dupont's class

The proving discourse of another French lesson, taught by Mr. Dupont, is another example of a reservoir-structure (See Figure 25). As in Pascal's class distinct argumentation chains (AS-1, AS-2, AS-3) in the first part are organised by an abduction. The first stream is a move forward (AS-1). It concludes that the inner quadrilateral in the proof diagram is a rhombus. Then, as in Pascal, an abduction establishes a datum as a target-conclusion. The datum 'The angle SRU is 90°' is then justified (in AS-2) and finally used to prove that the rhombus is a square (AS-3). The argumentation structure of the second part of the proving discourse is more straightforward.

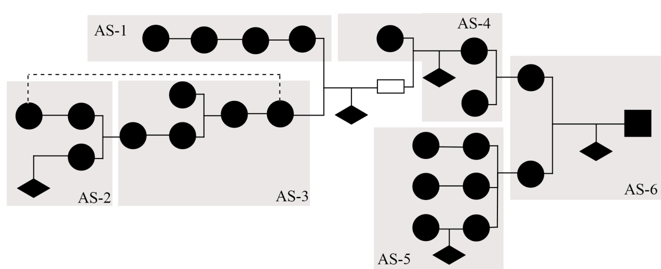


Figure 25. Overall argumentation structure in Dupont's class.

THE SPIRAL-STRUCTURE

We have applied Knipping's method of analysing classroom proving processes to data from Canadian classrooms, gathered as part of the RIDGE project (see http://www.acadiau.ca/~dreid/RIDGE/index.html). We have identified two argumentation structures in this context: the spiral-structure and the gathering-structure.

The spiral-structure shares some characteristics with Knipping's source-structure.

- Argumentation streams that do not connect to the main structure.
- Parallel arguments for the same conclusion.
- Argumentation steps that have more than one datum, each of which is the conclu-

sion of an argumentation stream.
– The presence of refutations in the argumentation structure.

Argumentation steps that lack explicit warrants or data occur less often than in the source-structure.

The main distinction between the spiral-structure and the source-structure is the location of the parallel arguments. Recall that in the source-structure the parallel arguments occur at the beginning of the process, and that later there is a funnelling into a single stream leading to the final conclusion. In the spiral-structure the final conclusion is repeatedly the target of parallel argumentation-streams. The conclusion is proven again and again, in different ways.

We have observed the spiral-structure in two cases, one of which we will use as an example here. This example comes from Ms. James' grade 9 (age 14–15 years) classroom in Canada. The class was trying to explain why two diagonals that are perpendicular and bisect each other define a rhombus. The students had discovered and verified empirically that the quadrilateral produced is a rhombus using dynamic geometry software and the proving process led by the teacher was framed as an attempt to explain this finding using triangle congruence properties.

Figure 26 shows the argumentation structure for this proving process. It displays several of the features discussed above:

– Argumentation streams that do not connect to the main structure (AS-C).
– Parallel arguments for the same conclusion (AS-B, AS-D, AS-E).
– Argumentation steps that have more than one datum, each of which is the conclusion of an argumentation stream (within AS-A and the final conclusions of AS-B and AS-E).
– The presence of refutations in the argumentation structure (AS-D).

In the argumentation structure three parallel arguments AS-B, AS-D, and AS-E lead to one conclusion, that the four sides are congruent, which acts as the datum for the final conclusion that the quadrilateral is a rhombus. In AS-B the congruency

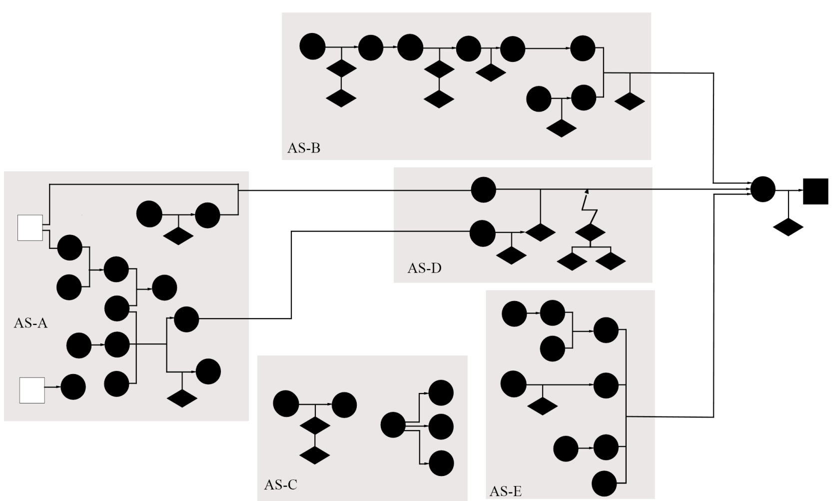


Figure 26. Argumentation structure in James' class for the rhombus proving process.

of the sides is shown by showing that the four triangles formed by the diagonals are congruent. In AS-D a student offers an alternative argument, based on the idea that the quadrilateral cannot be shown to be a square. This argument is listened to attentively by the teacher, who eventually refutes it. This is similar to the treatment of conjectures in the classes where the source-structure was observed. Finally, in AS-E the teacher offers an alternative argument based on using the Pythagorean Theorem instead of triangle congruency to establish that the four sides are equal. This argumentation stream is unusual because of the lack of warrants.

In the same classroom we observed a similar structure when the class was explaining why two congruent diagonals that bisect each other define a rectangle. These structures observed in Canada are similar in many ways to the source-structure, but they differ from it in an important way. In the source-structure the parallel arguments occur early in the proving process. The teacher invites input at this stage, but once the basis for the proof is established, the teacher guides the class to the conclusion through a structure that no longer has parallel arguments. In the spiral-structure, however, the conclusions to the parallel arguments are almost the final conclusion in the entire structure. In fact, the three parallel arguments could stand alone as proofs of the conclusion. Having proven the result in one way, the teacher goes back and proves it again and again. For this reason we describe these structures as *spiral*. Comparison of the argumentation structures in the lessons taught by Mr. Lüders, Ms. Nissen and Ms. James reveals some similarities, but also significant differences in the teaching approach in these contexts.

THE GATHERING-STRUCTURE

In the gathering-structure the argumentation includes the gathering of a large amount of data to support several related conclusions. New data is introduced as needed rather than being given initially, and the conclusions are also not specified in advance. Metaphorically, the class moves along, gathering interesting information as it goes. The gathering-structure differs from the source-structure and the spiral-structure in that it does not include parallel arguments for a single conclusion and argumenttation streams that do not connect to the main structure. However, like the source-structure there are many argumentation steps that lack explicit warrants or data. The gathering-structure is unlike the reservoir-structure as it includes refutations and lacks backwards reasoning.

Our example of a gathering structure (Figure 27) comes from the same Canadian grade 9 classroom as our examples of the spiral-structure, above. The students had explored independently, using dynamic geometry software, whether three side lengths given to them determined a unique triangle, and whether any three side lengths would determine a triangle. The argumentation structure represents the class discussion afterwards. In AS-A they conclude that the three given side lengths determine a unique triangle. Within this stream it is conjectured that more than one triangle is possible but this is refuted. This was intended to be the final conclusion,

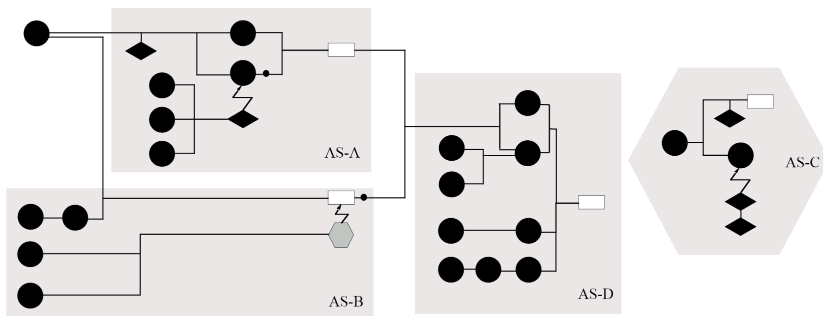


Figure 27. Argumentation structure in James' class for the side-side-side proving process.

so that it could be used as a basis for later arguments, but students' conjectures led to further discussions, involving additional empirical data and leading to other unanticipated conclusions. While the first conclusion was being discussed a conjecture was made that it is always possible to create a triangle given any three side lengths. AS-B is the argumentation stream resulting from the argument for and against this conjecture (marked by a white rectangle). The conjecture is refuted (by means of additional data gathered and brought in through AS-C) and its negation becomes the final conclusion for AS-B (indicated by the small black dot after the white rectangle). Having arrived at the two conclusions that three sides sometimes determine a unique triangle and sometimes do not determine a triangle at all, the students gathered more data related to the question of when three sides determine a triangle. In AS-D the students combine their conclusions from AS-A and AS-B with additional empirical evidence to conclude the triangle inequality: that a unique triangle is possible only if the sum of any two of the given side lengths is greater than the third.

Even though the same class is involved, this argumentation structure is quite different from the spiral-structure. In the gathering-structure shown in Figure 27 there are no parallel arguments (like AS-B, AS-D and AS-E in Figure 26) or disconnected streams (like AS-C in Figure 26) and very few warrants.

A similar gathering-structure was observed in another proving process in which the students were determining whether three given angle measures would determine a unique triangle, and whether any three angle measures would determine a triangle. In both of these proving processes the students' arguments were based largely on the empirical evidence offered by the dynamic geometry software. They lacked, at that point, sufficient prior knowledge to deduce their conclusions. We hypothesise that the lack of parallel structures is related to the use of empirical arguments, as for the most part evidence accumulates rather than arising in distinctly new ways.

IDEAS FOR RESEARCH

Above we have described four types of argumentation structures: the source-structure, the reservoir-structure, the spiral-structure and the gathering-structure. These have been observed in different contexts: in different countries and in classes focused on different topics. We believe analysis of argumentation structures can reveal different classroom cultures and approaches to teaching proof that follow their own peculiar rationales. Our examples here suggest that these classroom cultures may be influenced by the larger cultures in which they are embedded, but also by the nature of the mathematics being studied and the teacher's goals. This suggests two directions for future comparative research.

Sekiguchi (1991) and Herbst (2002ab) examine the origins and nature ofclassroom proving cultures in the US, typified by the form of proofs accepted: two-column proofs. This form was not the norm in the classrooms we described above. It would be interesting to see if there is an argumentation structure associated with the use of this form, or several related to the topic under consideration. Comparative analyses of this type can deepen our understanding of how classroom proving processes can be an obstacle or an opportunity for students to learn to reason mathematically and to engage in proving.

It would be interesting as well to examine another context: classrooms that explicitly espouse an inquiry mathematics approach. A comparison of the argumenttation structures that occur in such classrooms versus those that occur in more traditional classrooms would deepen our understanding of the differences inteaching and learning that occurs in these contexts.

As we mentioned in Chapter 8, Krummheuer (1995, 2007) sees individual learning in the classroom as dependent on the students' participation in "collective argumentation" (2007). Krummheuer understands participation in argumentations as "a pre-condition for the possibility to learn" (p. 62) and investigates classroom situations and their potential for learning. The classroom episodes discussed by Krummheuer often contain only "the minimal form of an argumentation" (Krummheuer, 1995, p. 243), consisting simply of data, warrant, and conclusion rather than the more complex streams and structures described here. This is not surprising given the grade level Krummheuer is investigating and that in most of the classrooms argumentation was not subject or goal of the lesson. The hypothesis that argumentations are "a pre-condition for the possibility to learn" would be interesting to investigate in more elaborated argumentations for which analysing and comparing argumentation structures provides a useful tool.

Further research on argumentation structures that build on Krummheuer's work together with Knipping's must take into account differences in their terminology, especially when writing in German. What we have called here a "step" (following Toulmin's usage) is referred to by Knipping (2003b) as a *Schritt* but by Krummheuer (2003, Krummheuer & Brandt, 2001) as a *Strang*. Krummheuer's "minimal form" differs from what we call a step as it must include the three elements, data, warrant, conclusion, while our step might have the data or warrant missing, and might include multiple sources of data. Knipping (2003b) uses Krummheuer's word for a step, *Strang*, to refer to what we have called here a "stream". Krummheuer does not describe anything exactly like a stream, but uses *Mehrgliedrige Argumentation*

(2001, p. 36) or “chain of argumentations” (2007, p. 65) to refer to a stream like AS-B in Figure 26 in which each datum is the conclusion of a prior step. Finally, our “parallel arguments” are called a *Parallelargumentation* by Knipping (2003b) and are similar to what Krummheuer calls an *Argumentationszyklus* (2003, p. 248) or cycles of argumentation (2007, p. 75).