GOOD-ENOUGH UNDERSTANDING: THEORISING ABOUT THE LEARNING OF COMPLEX IDEAS (PART 1)

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I, Vicki, am a classroom teacher and researcher. I returned to the fifth grade (ten- to eleven-year-old children) classroom in 1989, after more than a decade of teaching in a university faculty of education, in order to teach in the changing ecologies of classrooms (with problem-solving approaches in mathematics and literature-based approaches in reading) and to research from the 'inside'. Since 1996, David Reid and I have worked together on various aspects of the children's notions of proving and convincing. [1]

Vicki: On conversations that are springboards to thinking

In March 2000, while David and I were discussing our presentation and paper for the upcoming Canadian Mathematics Education Study Group (CMESG) meeting, I mentioned to David an article I had recently read, entitled, 'Good-enough reading: momentum and accuracy in the reading of complex fiction'. It interested me in and of itself, but also resonated with my personal experience as a learner of mathematics and as a reader of literature. I recall telling David (almost apologetically, because I was thinking, "Why would I be bringing literature into a discussion about fifth graders' work in mathematics?") that the ways the readers in the article moved through the complex readings made me think of how I navigated through my encounters with challenging material in mathematics.

Margaret Mackey, the author, wrote of readers' efforts to determine how to weight the significance of the items they were attending to. The ability to take note and then put aside, "to register that something may eventually be important without currently understanding how that importance might be manifested", was, she said, "a major element in good-enough reading" (1997, pp. 444-445). I was startled to hear David say that he knew the author of the article since she was a faculty member at the University of Alberta where he had pursued his doctoral work. There had been talk 'in the air' there about the idea of 'good enough' and he had already done some writing about his interpretation of 'good enough' in relation to enactivism (Reid, 1996).

Here then was an idea – the idea of 'good enough' – that resonated with each of us. On that day we began to discuss the idea of 'good enough' together, using that lens to look at some of the children's work from the 1999 year and continuing to explore that notion in the years which followed (2000, 2001).

In this article (the first of two parts), David and I will first tell you more about the 'good enough' idea, detailing some of Mackey's thinking and that of others whose thoughts have contributed to our working definition of 'good enough'. We will provide a little background about ourselves, the children, the setting, and the task so that you will know how our collaborative research arose and evolved and how our questions emerged, questions both about the mathematics and about how one comes to understand mathematics. Finally, we will offer two examples of how students operate with good-enough understandings. In Part 2, we offer a third example and use the examples to explore the idea of good-enough understanding further, building on ideas from Mackey's article and those of other authors.

Our work and this article are collaborations. However, we found it difficult to use 'we', or even 'I' (with narrators named) consistently in our writing here. The alternative, the use of a third-person point of view, made the telling too impersonal. We have therefore opted at times to use 'we', at times to present some of the narrative using Vicki as the first person 'I' voice and at other times to have David be the 'I' voice and do the telling.

On good enough and making do

In her article, Mackey (1997) proposed that learners work with 'good enough for the moment' ideas as placeholders, that is, when confronted by many complex ideas for the first time through learners seem to be wading in heavy water, making many tentative temporary decisions, keeping diverse and at times contradictory possibilities 'in the air' and waiting at times to the end to make sense of what has transpired. Mackey's work is on the reading of complex literature texts, but we will show here how much of what she says pertains as well to mathematics.

Mackey contends that "the ability to read further on the basis of a very imperfect understanding of the story" (p. 444) to that point in time is a vital and profoundly undervalued skill in complex reading. The same can be said for encounters with non-routine, complex problems in mathematics. Opting for a temporary decision which is 'good enough for the time being' is not only a good move, it is one which we make all the time when in the midst of learning. Newkirk (1984) has used the term "satisficing" (p. 759) to mean 'making do' on occasion by assigning a general meaning and then going on. Mathematical understanding is a process and as such is always contingent in nature.

David has spoken elsewhere (Reid, 1996) of conducting research in ways that acknowledge that final "theories-of" cannot be attained, but that good-enough "theories-for" are a worthy goal for research (p. 207). This view of research is based on the work of Varela *et al.* (1991), who describe the logic of natural selection as "proscriptive" and apply that logic to their theory of cognition. Proscriptive logic is based on what is forbidden, fatal or not good enough in some way. Natural selection, in this view, is not a matter of survival of the fittest (in which fitness is prescribed), but of extinction of the unfit. This idea, that nature selects not the best adaptations but allows all adaptations that are good enough, was pointed out as far back as 1956, by Simon: "organisms adapt well enough to 'satisfice', whereas they do not, in general, 'optimize'" (p. 129). Prescriptive logic allows only one correct way to proceed. Proscriptive logic broadens the range, allowing that there are a number of good-enough possibilities. Because of this openness, however, it is impossible to know if what seems sufficient at one point will remain so. Species behave 'as if' they are well adapted to their environments.

Similarly, students engaged in mathematical activity behave 'as if' their understandings are sufficient, as long as they do not fail in some way. Good-enough understanding means 'making do' on occasion, and moving on. In the everyday use of the term, some have equated the ideas of 'good enough' and 'making do' with laziness. However, we contend that good enough is the best we can do when doing our best, that is, when putting in maximum effort. In evolution and, we believe, in mathematical activity, trying to optimise can be counterproductive as energy is wasted trying to achieve an impossible goal.

Vicki: The classroom setting and the task

I realise more and more that my past life experience as a child and young adult, as one who feared mathematics and avoided it as soon as I completed high school, has shaped my vision for how it might be different for the children with whom I now work. What I desired for my students in the 1990s in their encounters with mathematics differed from what I myself knew as student or as a teacher of fifth graders in the late 1960s. Prevalent among my goals were having the students have a comfort zone, feel a sense of agency and ownership of ideas, feel empowered to question the text, have ample time to explore, to explain their thinking, justify their thinking and respond to others' explanations of their thinking. As the children and I worked together, I looked at consequences of that participation for developing understanding and identity (see for example, Zack and Graves, 2001) and liked what I saw. I in turn provided the children with yet more time for conversations.

Non-routine problem solving is central to the mathematics curriculum at all levels in our school. Mathematics is officially scheduled for forty-five minutes each day (but twice a week I extended my classroom mathematics time to ninety minutes). Extended investigations of non-routine problems take up the entire lesson three times a week. The children are given time to experiment with, think through, discuss and refine their understandings. They are expected to take responsibility for revisiting ideas and at times they pursue their own questions. Each year's cohort is usually about twenty-five students, but mathematics in done as a heterogeneously-grouped half-class of twelve or thirteen. When working on a problem, the children first work in pairs (or a group of three), then come together in a group of four or five to compare solutions and discuss them, finally reporting to the half-class with more discussion. In this way, each problem or task is examined on four or more separate occasions in multiple contexts. (Other episodes and interpretations involving some of the same children can be found in Reid, 2001, 2002a; Zack, 1997, 1999.)

The task

The task as I presented it began with me asking the children how many squares of different sizes there were in a four by four grid (first assigned in May, 1994):

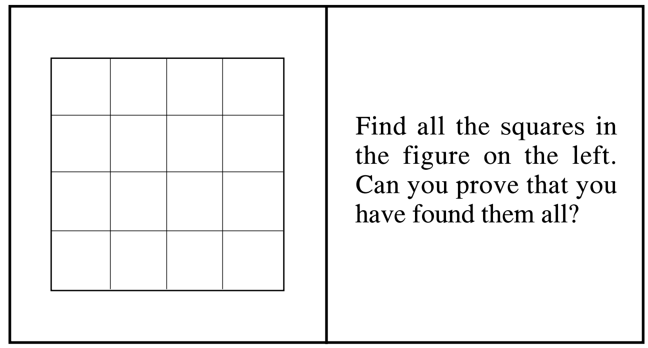


Figure 1: The task from 1994

The children discerned patterns (Zack, 1997) and I built upon the children's work. I expanded the process each year, adding extensions in order to see whether and how the children could see and generalise various patterns, primarily the sum of squares. What if it were a 5 by 5 [grid]? (added May 1995). What if it were a 10 by 10? A 60 by 60? (added May 1996). Originally, my expectation was that some children might see the pattern of the sum of the squares, and express their hypothesis that the pattern would continue, i.e. $1^2 + 2^2 + 3^2 + 4^2 + 5^2$ . . . Thus, if the children were able to generalise and say that the pattern would continue in the same way, that was good enough; indeed I felt that was sufficient for fifth graders. When I posed the "What if it were a 60 by 60 square?" question in 1996, what I expected the children to say was 'the pattern just continues'. I did not expect, nor did I want, the children to work out the actual numerical answer. They ignored me.

This unexpected development led to a number of surprises. Some children insisted that there must of course be an 'easier' way than adding square numbers all the way up to 60^2, but I was not aware in 1995/6 that it was impossible for the children to construct an algebraic expression for this generalisation. I could not derive the expression either, so I sought out other resources, people as well as book sources.

I was pleased to find and bring to the children the formula Johnston Anderson (1996) reported: $n(n + 1)(2n + 1) \div 6$, one which Anderson himself called a "non-obvious expression". My expectation was that all the children would use it and see that it fitted all of the examples, which they had calculated concretely. That was good enough for me. I did not know *why* the expression worked; I only knew *that* it did. However, the students wanted to know why it worked as it did (Zack, 1997). They also wanted to know how anyone could come up with that expression. It did not fit the kind of algebraic expression which some of them had been able to construct in other instances, where there was a meaningful connection and transition between the concrete examples and the general algebraic expression (Zack and Graves, 2001). I went to a professor in the mathematics education department in one of the local universities seeking an explanation to bring back to the students. I was told that ten- and eleven-year-old students could not construct or understand the mathematical approach to the construction of Anderson's algebraic expression given the state of their current mathematical knowledge. Indeed, neither could I.

It seemed at that time a dead end. And so, in 1997, a paper I wrote that year (Zack, 1997) ended with a list of the children's emergent definitions of what they felt proof ought to be, among them that the proof must make sense and that the person presenting it must say why it works. When asked, "What do you think of Johnston Anderson's rule?", the children responded that explanations and proofs should make sense. Ross, for example, said that Anderson's rule was "brilliant, but he should explain how it works". Lew said that "if the Johnston rule had evidence, if Johnston himself explained why it worked, it would be more convincing". And Rina felt that Anderson's expression was:

> a great way to figure out the problem but it doesn't make sense. … I think a mathematical proof is when you say why it works and if it works for everything show why. (Zack, 1997, p. 297) [2]

Perhaps due to my classroom emphasis on explaining oneself, the children pushed to know the whys and hows. All but one of the children polled (of a total of 10 that year) stated unequivocally that a proof ought to explain. It was at this point that I enlisted David's help and involvement.

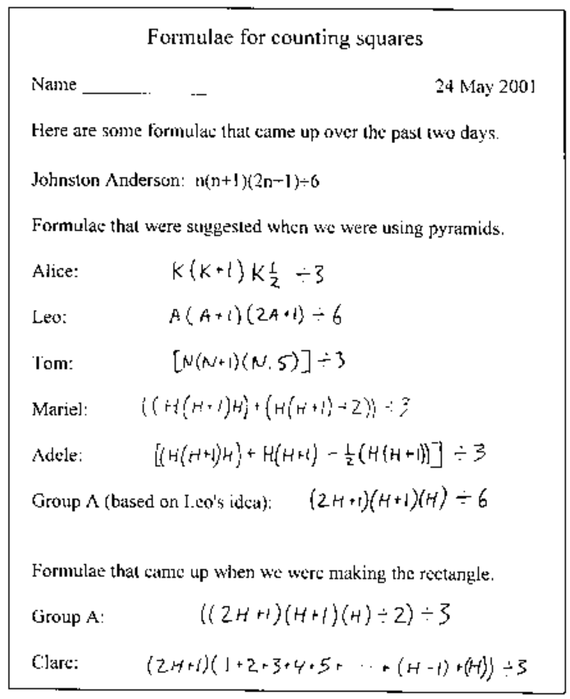
David: My role as teacher + researcher + mathematician in Vicki's classroom

I am a university mathematics educator and researcher, and have been working with Vicki over a number of years (in particular 1995-1996, and 1998-2001). When Vicki challenged me in 1996 to develop a way of explaining an algebraic formula for the sum of square numbers to a class of fifth-grade students, this appealed to me in several ways. As a researcher of the learning of mathematical proof I am interested in the role of different kinds of reasoning in explanations, and especially in the extent to which informal explanations that follow the logical structure of mathematical proofs are acceptable to students. I am also interested in exploring the extent to which Bruner's (1960) hypothesis "that any subject can be taught effectively in some intellectually honest form to any child at any stage of development" (p. 33) can actually be realised in practice. My research also includes examination of my role as a teacher + researcher (see for example Reid and Brown, 1999) and in this case as teacher + researcher + mathematician.

After Vicki's challenge I explored and deliberated over various approaches, at times alone and at times consulting with other mathematicians and mathematics educators. I decided on three possibilities and at various times showed and discussed these with students from Vicki's classes, each time meeting with some measure of success and a number of unanswered questions (see Reid, 2002a for full descriptions of the three approaches). The episodes used as examples here come from my later efforts to provide a formulation that met both the students' and my own criteria for an explanation.

Vicki's class works on the *Count the squares* problem in late April each year. I made my guest appearances at the end of May, after the children had arrived at the point of wanting a formula, and when they were dissatisfied after Vicki gave them Johnston Anderson's formula without being able to provide an explanation for why it worked or where it had come from.

Normally, Vicki guides the discussion at the conclusion of the children's work together, but the episodes described here began instead with an extended presentation by me as the focal teacher for that week. I presented my explanations, two visual proofs, and engaged the students in discussions in small groups or half-class groupings over the first two or three days. What happened next was different in the two years from which we have taken the episodes. In 1999, our focus was more on the students' conceptions of what a proof is in mathematics. We followed up my presentations by asking some questions related to that focus (see Reid 2002b for some analysis of those results). In 2001, the students worked in pairs, then in groups of four, to reconstruct what I had presented. On page one of the worksheet (see Figure 2 [3]) that the students used, I had collected together formulae for counting squares.


Formulae for counting squares

Name ______ 24 May 2001

Here are some formulae that came up over the past two days.

Johnston Anderson: n(n+1)(2n−1)÷6

Formulae that were suggested when we were using pyramids.

Alice: K(K+1)K½ ÷3

Leo: A(A+1)(2A+1) ÷ 6

Tom: [N(N+1)(N.5)] ÷ 3

Mariel: ((H(H+1)H) + (H(H+1) ÷ 2)) ÷ 3

Adele: [(H(H+1)H) + H(H+1) − ½(H(H+1))] ÷ 3

Group A (based on Leo's idea): (2H+1)(H+1)(H) ÷ 6

Formulae that came up when we were making the rectangle.

Group A: ((2H+1)(H+1)(H) ÷ 2) ÷ 3

Clare: (2H+1)(1+2+3+4+5+ … +(H−1)+(H)) ÷ 3

Figure 2: Page 1 of worksheet: a collection of formulae for counting squares

On page two of the worksheet were a set of questions.

> Copy the formula you feel most comfortable with here.
>
> Explain each part of the formula.
>
> Why is there (H + 1) in most of the formulae?
>
> In Leo's formula why is there (2A + 1)?
>
> In Tom's formula why is there (N.5)?

> Why is there ÷ 3 in some of the formulae?
>
> Why is there ÷ 6 in some of the formulae?
>
> Are some of the formulae the same? Which ones?

Answering these questions prompted them to revisit certain components and to check their understanding. The students were videotaped throughout their pair, small-group, and half-class discussions.

We will deal here with the students' understandings of three critical ideas in my presentation of the tri-pyramid proof (the first of the two proofs I presented to the students [4]). In this proof, three pyramids model the sum of the first n square numbers. They are assembled into a three-dimensional object (see Figure 3).

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{3}n(n+1)(n+\frac{1}{2})$$

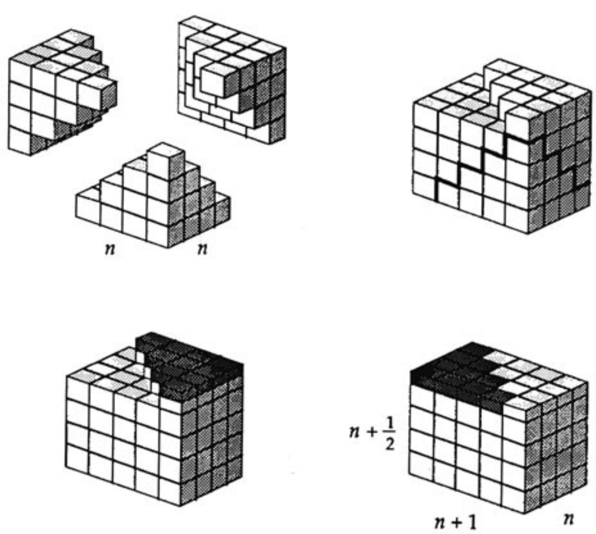


Figure 3: The tri-pyramid proof (from Nelsen, 1993)

This object was called a number of names by me and by the students. Here we will use the term 'quasi-box', unless we are quoting a student's usage of another term (as for example, Leo's coining of the term "almost cube"). The dimensions of the quasi-box are n, n + 1 and n + 1/2. This leads to the formula for the number of cubes making up one pyramid (and hence the sum of the first n square numbers) being, for example: $n(n + 1)(n + 1/2) \div 3$.

There were variations from year to year and from group to group in my presentation of this proof. All my presentations followed approximately this outline: I first built a four-layer pyramid and explained the connection of the four-layer pyramid to 1 + 4 + 9 + 16. I then assembled three pyramids to make a quasi-box. I found the dimensions of the quasi-box and recorded the product 4 × 5 × 4.5. I repeated this process with a five-layer pyramid, and then invited the students to consider an imaginary larger case and a general case.

Part of my conversation with the children included a discussion of my own understanding of the tri-pyramid proof. I related having seen it (in Nelsen, 1993), spending two years trying to understand it, and finally being able to present it to the class. In telling this story, I modelled mathematical inquiry as a process that is more than just looking up the right answer in a book, as one that also includes a quest for understanding of the answer. The teaching also included sharing with Vicki my efforts over time to arrive at an explanation that could be understood by her students, my past attempts with other explanations and our excitement at the students' understanding and working with the algebraic expressions. Our emotions as researchers in mathematics education blended into our emotions as teachers, and became for the children something that was about people, namely about Vicki and myself, but also at the same time about teaching and research.

Crucial ideas

There are a number of crucial ideas (see Figure 4) that must be understood in order to understand my explanation as a whole. In another paper (Zack and Reid, 2001), we describe these crucial ideas. Here and in Part 2, we present episodes that illustrate the children's good-enough understanding related to three of them: the division by 3, the half-layer, and (in Part 2) the factor 2A + 1. Not all the children understood all these ideas, and those that did understand them did not all understand them at the same time. Nonetheless, they were able to attain partial understandings of the explanation that were good enough to support their continuing engagement in the mathematical activity of the class. We see this situation as similar to that of readers of fiction, such as those described by Mackey:

> What the readers supplied for themselves was good enough. They would have liked more information but

	Crucial ideas
1	To count squares you add 1 + 4 + 9 +16 + ... These numbers are SQUARE numbers in the sense of being N x N.
2	The NUMBER of blocks in a pyramid of height N is the SAME AS the NUMBER of squares in a N x N grid (1 + 4 + 9 + ... + N x N)
3	Assembling three pyramids always produces a similar three dimensional object, the "quasi-box".
4	One face of the quasi-box is made up of an entire N x N square, plus the edge of another one, forming an N x (N + 1) rectangle.
5	The top layer contains HALF as many blocks as the other layers, which is the same as a FULL layer of HALF blocks.
6	The number of squares in the top layer can be found by dividing the number of blocks in a full layer by 2.
7	Arrays: That a 3-dimensional box is composed of A x B x C little cubes.
8	If you use three pyramids you have to divide by 3 later. If you use six pyramids you have to divide by 6 later.
9	Two quasi-boxes can be put together to make a box, with a longest dimension of 2A + 1.
10	You can use a letter to stand for a variable in a formula.

Figure 4: Crucial ideas that children might not understand but hold and wait for understanding

not enough to stop them in their forward progress through the book. (1997, p. 448)

The *first* example follows and is that of Maya (from the 1998-1999 class) and her understanding of the reason for the division by 3. Because three pyramids are combined, it is necessary, later, to divide the number of cubes used by 3. This example illustrates the balance/conflict that Mackey calls *contingency versus coherence*. Mackey suggests that synthesis occurs only at the end of the reading or discussion. During the first time through the process, ideas are tentative, contingent – a swarm of impressions and associations. Mackey's emphasis is upon the untidy and inevitably partial nature of the readers' accommodations to the text on their *first* readings of it. We would add that revisiting also renders more and yet understanding is always in transition.

David: Maya and the idea of dividing by three

My previous experiences, using other concrete and numerical explanations of formulae for the sum of square numbers, led me to expect the division by three to be a source of confusion (see Reid, 2002a). However, when working with the first group in 1999, I put no special emphasis on my use of three pyramids to form the quasi-box. I said only that using more pyramids made it easier to determine the number of cubes involved. After we had worked out the 4 by 4 case, a comment from one student led me to believe that the division by three was not going to be a big issue after all. The story of that student's shifting understanding follows.

I was struggling to understand a comment from Tessa about the need to count the interior cubes of the quasi-box when Maya raised her hand with a gasp of excitement. [5]

Maya: I was going to say maybe you divide by three because there are three thingies there, but then I realised that would only give you the number of little blocks there are, not the number of squares there are.

DAR: If we divided the number of blocks in this big thing by three we would get the number of blocks in just one of the pyramids?

At this point, the important issue seemed to be that Maya did not understand that "the number of little blocks there are" is actually the same as "the number of squares there are" in the *Count the squares* problem. I assumed her insight into the need to divide by three could be kept in suspension until I needed it later. But when I tried to return to her comment, after working out the three dimensions for the 5 by 5 and 6 by 6 cases, she seemed to have forgotten about it.

DAR: Now somebody said something about there being too many blocks in here – because we used three pyramids to do it? – who was it, who told me about that?

Tessa: I might have, but I don't know -

VZ: I think it was Maya.

Maya: [looking surprised and confused] What did I say?

Tessa still had some questions about how 'knowing the three dimensions of the box' was related to 'the total number of cubes in the box' (another crucial idea). I delayed the issue of dividing by three again, but this time with an awareness that it was not going to be as easily dealt with as I had hoped. I had them work out the three dimensions for the 10 by 10 case to see if they had generalised the structure of the three factors (rather than building each one up from the previous case) and then asked them to compare their answer (1155) with what they knew was the right answer for the 10 by 10 *Count the squares* problem (385).

DAR: Right. So, one thousand, one hundred and fifty-five is what you get if you multiply those three numbers. Is that how many squares there are in a ten by ten?

Several voices: No.

DAR: So what's wrong?

Shelley: I think you divide – by three –

(Shelley and Gino had worked on the explanation with me in a small group before I met with the half-class.)

DAR: Shelley thinks we ought to divide by three.

Shelley: No I forget. We were supposed to divide it.

Gino: I do. I think you have to divide by three also.

Maya: Gino, you know the answer.

Gino: But Shelley just said -

Shelley: No but I -

DAR: OK.

Maya: But I said that before.

DAR: Could we have somebody other than my helpers here, suggest a reason why we might want to divide by three – Mona?

Mona: Because there's three numbers.

DAR: Because there's three numbers. That's a good reason.

Mona: I guess.

DAR: OK –it's not a great reason, but it's a good reason. Sorry, I've forgotten your name.

Elaine: Because there's three of those triangle thingies in there.

Maya: But then why wouldn't you divide it by

	three and a half, because there's a half?
DAR:	OK, how many triangles did we put together to make this thing?
Several voices:	Three.
Maya:	And then there's the half.
DAR:	And in each triangle, in this triangle, in this pyramid, in this whatever it's called – there are there are the same number of blocks as there are in the five by five, squares in the five by five – right? That's how we put it together, we did one plus four plus nine plus sixteen plus twenty-five. So then we put together three of them and suddenly we had three times too many, – so that would be a good reason to divide by three if you've got three times too many of something. What do you get if you divide one thousand, one hundred and fifty-five by three?
Several voices:	Three hundred and eighty-five.

When Shelley and Gino (who had seen the explanation before) said it was necessary to divide by three because there were three pyramids, Maya seemed to remember her previous comments, "But I said that before", but then she objected "But then why wouldn't you divide it by three and a half, because there's a half?" suggesting she had a different understanding. This response suggests that she had picked up on my calling attention to the importance of the half-layer earlier in the explanation and so when I asked for an explanation of the division by three, she drew on the element I had stressed, even though there is no obvious (to me) connection between the half-layer and dividing by three. We will be examining a different student's encounter with the half-layer shortly.

Six months later, Vicki asked all of the former fifth-grade students, now in sixth grade, about the formula. At this time, Maya had no trouble explaining every part of the expression, including relating the division by 3 to the use of three pyramids. As far as we know, she had not thought about the problem since the class had worked on it, yet her understanding that was good-enough at that time had evolved into an understanding that we see as deeper and more complete.

Vicki and David: Commenting on Maya's understanding

Maya worked to understand what David was saying. She raised the need to divide by three very early, before anyone had noticed that it was an issue, although she seemed to forget about it later. She also objected if reasons seemed to her to be inadequate (e.g. "But then why wouldn't you divide it by three and a half, because there's a half?"). At times she demanded that the explanation be *coherent*, but at other times she was willing to let things go without questioning them, to leave them *contingent*. This is one way in which understandings can be good enough. Coherence is not required throughout. In fact in Maya's case it may be that some parts of the explanation were not coherent until some time later.

Maya's case may be part of a general pattern of noticing and marking (Mason, 1994). When David first used three pyramids to build the quasi-box, the students noticed that he had done so. That is, they could recognise this fact when it was referred to later. Most of them did not *mark* it however, so they could not bring it to mind as a possible explanation for the division by three when he asked them why it was necessary. In Maya's case, she may not have marked her own comment in spite of David repeating it at the time, so when we referred to it later she did not know what we were talking about. Later, however, she recognised what Shelley and Gino were saying as the same as what she had said about dividing by three, although her retrospective marking of that comment did not include her explanation relating to the pyramids.

To understand the explanation it is not necessary to mark (that is, be able to explain) the use of three pyramids when it occurs, but it is important to notice it, so that the division by three can be explained by it later. That the division by three was recognised as an important element in the explanation is suggested by the children's responses to the question, "Which parts of the rule/formula do you understand?", when it was asked the following day. Twelve of the twenty-five children who had seen David's explanation remarked on the importance of the division by three, and four of those went as far as to mention the connection to the three pyramids.

David: Leo and the idea of the half-layer

Combining three pyramids produces a quasi-box, which is almost regular except for a staircase-shaped half-layer on the top. Our second example of a good-enough understanding is the case of Leo's understanding of this half-layer. It illustrates contingency versus coherence, the acceptance of partial information and the tension between the need for momentum and the need for accuracy.

Leo's first encounter with the idea of the half-layer occurred on May 22, 2001 in the context of my presentation to a half-class group of 13 students. Leo's second encounter with the idea came two days later. The children were asked to talk together in pairs and explain to each other parts of the formula developed by members of the class based on my visual explanation (see Figure 3). Leo and his partner Seth then joined Alice and Jane to discuss the concepts once more. Six months later (December 6) Vicki asked Leo and Seth to review their written answers from May 24, so Leo had a fourth encounter with the concept.

When I presented my visual explanation to the large group (of 13 students) in 2001, Alice was the first to outline the role of the half. Her approach in the 4 by 4 case was to multiply 20 by 4, and then add half of 20. Leo then gave his version. He seemed to see the 4 1/2 as a single number, "Four by five and then there's a half one so it's four and a half".

When the students were asked to write a formula, Leo wrote three algebraic expressions on his sheet. (See [3].)

$X(X + 1)(X + 1/2) \div 3$	1/4 pyramid
$X(X + 1)(X + 1/2)$	not really a cube
$X(X + 1)(X + 1/2) \times 2$	cubic rectangle

In all of these he uses $(X + 1/2)$, which suggests that the verbal discussion of the half-layer has been understood by him is a way that is good enough for him to symbolise it in a precise way.

Two days later, Leo's understanding was less secure. He was working with his partner Seth to interpret the formula $K(K + 1)(K\ 1/2) \div 3$, which was ascribed to Alice based on her work in the class group on May 22. Leo interpreted (K 1/2) as meaning K + (1/2 K). It is possible that the notation confused him, as his own notation (X + 1/2) is more clear. It is also possible that he had simply forgotten where that part of the formula came from. When calculating for K = 5 he claimed "Five and a half is seven point five". Later he seems to have understood that (N.5) is the same as (K 1/2) and that both refer to the half-layer, but he still calculated "N and a half, 4, 6" when N = 4. Shortly after, however, he seems to realise his error. At first he repeated "Four and a half is six" but then he looked thoughtful and exclaimed "Oh!" and said, with emphasis "times four point five, Seth". Leo also asked, "Why do you multiply point 5?", but let that question drop without pursuing it.

An intervention by Vicki a few minutes later provided an alternative interpretation in terms of the dimensions of the "almost cube". After this Leo seemed to have no trouble explaining to others the meaning of (K 1/2) and (N.5) in terms of the height of the "almost cube" being halfway between K and K + 1. That perspective was good enough for the moment. Leo left aside for the time being the aspect of multiplying by the 'half'. It is important to note that Leo was aware that the 'half' was giving him difficulty. He knew that the answer for the 4 by 4 ought to be 30, and for the 5 by 5 ought to be 55, and when he arrived at strange answers he knew something was not coming out right. Even when he realised that it must be "times four point five" not four and half of four, he was not sure about what to multiply, saying to Seth "half of what?"

Time and again Leo bumped up against the 'half', knew some of it or much of it was not clear or not 'coming out right', but each time after some consideration he left the question unresolved in order to continue with other aspects. He opted in this case for contingency over coherence. It was only later that afternoon, during the group of four discussion, that it was clear that Leo had become comfortable with the half-layer and the use of 1/2 and .5 in the formulae. That happened in large part due to a conversation between two of his partners in the group of four, Alice and Seth.

It was during that discussion that Seth's uncertainty about the half also became evident. Seth asked Alice and Jane about 'half' in relation to Alice's formula. "Doesn't that mean minus half of the whole thing?" Alice said, "No, it's half of K times K + 1, five times four is twenty so it's ten". Alice never used the term 'layer', nor stated explicitly that you are multiplying to find out the answer to how many cubes there are in half of one layer. However, Alice's response to Seth here may well have helped Leo come to an understanding of how the 'half' works, because although Leo subsequently launched into an account of how working with a half was challenging in this instance, he also stated, "I understand it now".

Vicki and David: Commenting on Leo's understanding of the half-layer

Leo's original understanding of the half-layer was good enough for him to symbolise the factor in the formula as (X + 1/2). However, it was not good enough to allow him to interpret the formulae of others later. He was aware of this problem in his understanding and this contributed to his decision to work instead with his own formula: $A(A+1)(2A+1)/6$. The fact that the explanation they had been given gave rise to multiple formulae constructed by diverse classmates turned out to be an advantage here, as an understanding which was not good enough to give Leo confidence using some formulae was good enough for him to use another one, namely his own as he was thereby able to circumvent having to work with the half.

Leo did not stop and ask explicitly about the half even though he knew it was not coming out right. As we have said, Mackey suggested that even though her readers might have liked more information it was not enough to stop them in their forward progress. Leo too might have wanted to push on since there were many items to think about and discuss. Mackey contends that there is a tension between the need for momentum and the need for accuracy and accountability (1997, p. 457). Moving too slowly and trying methodically to fill in too many of the gaps the first time through can result in loss of rhythm. On the other hand, making decisions without being accountable to the core ideas is risky in that it may result in one proceeding down a dead end or arriving at the wrong conclusions (p. 431). Leo realised that he did not understand how the half worked in the formula, but perhaps decided to let it go in the hope that further along the way it would come clear. Leo's decision not to stop and work on solving the riddle of the half resembles a reader "moving forward with unanswered questions placed in suspension" (p. 449). Of course, when Leo heard Alice's explanation to Seth, it provided him with the answers he needed.

Vicki and David: Looking ahead and back

The episodes of Maya and Leo above show students who were able to continue to work on a problem even though they did not fully understand important aspects of it. The students do not understand some of the crucial ideas when they are first presented, at times they hold them in abeyance and wait, at other times they ask questions. As in the case with Mackey's students, these children create, extend, evaluate, and sometimes discard their initial assumptions and questions (cf. Mackey, 1997, p. 430). There are messy, loose ends during the conversations, with tentative and unfinished components in the learners' responses. Under the pressure to progress and try to get a picture of the whole, learners often resort to making good-enough decisions, contingent and temporary, based upon less than complete information (Mackey, 1997, p. 429).

Here Leo and Maya understood less than we thought they did when we first observed them in class, raising the questions of whether and how teachers can know what their students understand. In the second part of this article we will examine a third case, where Leo had different understandings of a crucial idea on different occasions, and how those understandings continued to be good enough.

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Notes

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[2] Vicki: In the case of this episode, the students not only asked provocative questions, such as, "How does this equation work?" but related in a human way to the mathematician. As you see in the case of this study, they called him Johnston. Note the personal tone: "if the Johnston rule had evidence, if Johnston himself explained how it worked, it would be more convincing [. . .]" ? and "Johnston Anderson's rule is brilliant, but he should explain how it works". These children see mathematics as a human endeavour. In encouraging the link, in 2001 I nudged the children to contact Johnston Anderson directly via e-mail to pose any questions or thoughts they had, as he had given us permission to do. Actually, none of the children did pursue the contact. However, the students did have access to David Reid, who was the mathematician on the inside. During that week in May, David spoke with the children, not only about how hard he worked to understand the proof in the Nelsen (1993) book, and arrive at a presentation of the proof that might make sense to them; he also interacted with the children as they worked in pairs and small groups of four. At times there was even a blending of the two, the mathematician from the outside (Johnston Anderson) and the one from the inside (David Reid). For example, one of the children, Jane, called out to 'Johnston' at one point, ready to ask her question, and corrected herself, giggling, by saying, "Oh, no, it's David". In my own school experience, the examples from the textbook came from some place on high; in my life experience as a young student no one questioned what the textbook or the answers at the back said. There was no human entity attached to the formulation of mathematics tasks.

[3] A note on notation: in the worksheet (Figure 2) and throughout this article we have retained the notations the children originated. Different letters (A, K, N, and X) were used by them in developing their formulae. This seemed not to cause any confusion for the children. They also used (K 1/2) and (N.5) to represent n + 1/2. Equivalent formulae sometimes appear in different notations.

[4] David: The day after I explained the formula using the tri-pyramid proof, I offered the students an alternative explanation. Although no one in these classes seemed to have a problem with a three-dimensional object (the quasi-box) representing the sum of two-dimensional objects (squares), I wanted to offer an explanation that avoided this possible problem. The alternative explanation is also based on a proof in Nelsen (1993). In it two sets of squares are laid out to form an irregular staircase, and a third set of squares is disassembled into 'sticks' whose lengths are the odd numbers that constitute the squares. These sticks fill in the spaces in the staircase to form a rectangle that measures $2A + 1$ on one side, and $(1 + 2 + 3 +\ldots+ A)$ on the other side. Previously we had established that $(1 + 2 + 3 +\ldots+ A)$ is $A (A + 1)/2$, so this explanation gives the formula: $(2A + 1)(A (A + 1)/2)/3$. This explanation reinforced both the occurrence of the $2A + 1$ factor (while explaining it differently) and the Gauss formula for the sum of positive integers.

[5] Note that we have used DAR to indicate David, the teacher, and VZ for Vicki, the teacher, so that the reader can more readily distinguish between the teacher talk and the student talk. In our transcripts, an en-dash indicates a short pause, a hyphen indicates interrupted speech and an ellipsis indicates omitted speech.

References

Anderson, J. (1996) 'The place of proof in school mathematics', *Mathematics Teaching* **155**, 33-39.

Bruner, J. (1960) *The process of education*, Cambridge, MA, Harvard University Press.

Mackey, M. (1997) 'Good-enough reading: momentum and accuracy in the reading of complex fiction', *Research in the Teaching of English* **31**(4), 428-458.

Mason, J. (1994) *Researching from the inside in mathematics education: locating an I-you relationship*, Milton Keynes, Centre for Mathematics Education, Open University. (An expanded version of: 'Researching from the inside in mathematics education: locating an I-you relationship', in Ponte, J. and Matos, J. (eds), *Proceedings of the Eighteenth Annual Conference of the International Group for the Psychology of Mathematics Education*, Lisbon, Portugal, **1**, pp. 176-193.)

Nelsen, R. (1993) *Proofs without words: exercises in visual thinking*, Washington, DC, Mathematical Association of America.

Newkirk, T. (1984) 'Looking for trouble: a way to unmask our readings', *College English* **46**(6), 756-766.

Reid, D. A. (1996) 'Enactivism as a methodology', in Puig, L. and Gutiérrez, A. (eds), *Proceedings of the Twentieth Annual Conference of the International Group for the Psychology of Mathematics Education*, Valencia, Spain, **4**, pp. 203-209.

Reid, D. A. (2001) 'Conjectures and refutations in grade 5 mathematics', *Journal for Research in Mathematics Education* **33**(1), 5-29.

Reid, D. A. (2002a) 'Elements in accepting an explanation', *Journal of Mathematical Behavior* **20**(4), 527-547.

Reid, D. A. (2002b) 'What is proof?', *La Lettre de la Preuve: international newsletter on the teaching and learning of mathematical proof*, Summer 2002, http://www.lettredelapreuve.it/Newsletter/02Ete/WhatIsProof.pdf

Reid, D. A. and Brown, L. (1999) 'Fork in the road', *For the Learning of Mathematics* **19**(3), 15-22.

Simon, H. (1956) 'Rational choice and the structure of the environment', *Psychological Review* **63**(2), 129-138.

Varela, F., Thompson, E. and Rosch, E. (1991) *The embodied mind: cognitive science and human experience*, Cambridge, MA, The MIT Press.

Zack, V. (1997) '"You have to prove us wrong": proof at the elementary school level', in Pehkonen, E. (ed.), *Proceedings of the Twenty-First Conference of the International Group for the Psychology of Mathematics Education*, Lahti, Finland, **4**, pp. 291-298.

Zack, V. (1999) 'Everyday and mathematical language in children's argumentation about proof', *Educational Review* **51**(2), special issue, Burton, L. (ed.), 'The culture of the mathematics classroom', pp. 129-146.

Zack, V. and Graves, B. (2001) 'Making mathematical meaning through dialogue: "Once you think of it, the z minus three seems pretty weird."', in Kieran, C., Forman, E. and Sfard, A. (eds), 'Bridging the individual and the social: discursive approaches to research in mathematics education', Special Issue, *Educational Studies in Mathematics* **46**, 229-271.

Zack, V. and Reid, D. A. (2001) 'A proof ought to explain: a classroom teacher-researcher, a mathematics educator, and three cohorts of fifth graders seek to make meaning of a non-obvious algebraic expression', Topic Group report, in Simmt, E., Davis, B. and McLoughlin, J. (eds), *Proceedings of the 2000 Annual Meeting of the Canadian Mathematics Education Study Group*, Edmonton, AB, CMESG/GCEDM, pp. 95-102.